On the four-photon decay of the neutral pion

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The dominant contribution to the $\pi^0 \to 4\gamma$ branching ratio, coming from the purely electromagnetic photon-splitting graph, is calculated. The result $Br(\pi^0 \to 4\gamma) \simeq (2.6 \pm 0.1) \cdot 10^{-11}$ is about three orders of magnitude below the present experimental limit. © *1995 American Institute of Physics.*

Although the C-violating decay $\pi^0 \to 3\gamma$ is expected to have an extremely small branching ratio,[1] beyond the reach of any present or future experimental facilities, its experimental study has attracted considerable attention[2] because any observed anomaly in this process would be a clear signal of a new physics.

Any $\pi^0 \to 3\gamma$ searching experiment has, as a by-product, information about the allowed decay $\pi^0 \to 4\gamma$, which is a potential background for $\pi^0 \to 3\gamma$. The experimental upper limit on the branching ratio $Br(\pi^0 \to 4\gamma)$ was gradually improved[3–5] and was lowered to $2 \cdot 10^{-8}$ in Ref. 2. Some theoretical estimates for $Br(\pi^0 \to 4\gamma)$ can be found in the literature,[6–8] with rather broad ranges from 10^{-9} to 10^{-16}. In our opinion, the results of Ref. 8 are the most reliable, the authors giving the most thorough investigation of the subject.

As argued in Ref. 8, the dominant contribution to the $\pi_0 \to 4\gamma$ branching ratio is expected to come from the purely electromagnetic photon-splitting graph of Fig. 1, contributions from any hadronic intermediate states being less significant, especially for PCAC-satisfying models. The calculation of this contribution has not been done, to the best of our knowledge, and will be performed in the present note.

Using the standard covariant phase-space calculation technique[9] and factoring out some numerical constants from the decay amplitude, we can write

$$Br(\pi^0 \to 4\gamma) \simeq \frac{\Gamma(\pi^0 \to 4\gamma)}{\Gamma(\pi^0 \to 2\gamma)} = \frac{1}{6\pi}\left(\frac{\alpha}{8\pi}\right)^4 R, \tag{1}$$

where

$$R = \int_0^1 ds_1 \int_0^{s_1} ds_2 \int_{s_2/s_1}^{1-s_1+s_2} \frac{du_1}{\sqrt{\lambda(1,s_2,s_2'}}$$

$$\times \int_{u_2^-}^{u_2^+} du_2 \int_{-1}^{1} \frac{d\zeta}{\sqrt{1-\zeta^2}} F(s_1,s_2,u_1,u_2,t_2(\zeta)). \tag{2}$$

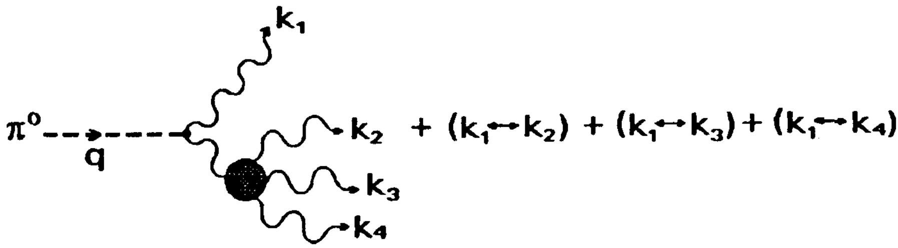

FIG. 1. The electromagnetic photon-splitting graph.

In (2) we have introduced a dimensionless version of Kumar's invariant variables[9]:

$$s_1=\frac{1}{m^2}(q-k_1)^2,\quad s_2=\frac{1}{m^2}(q-k_1-k_2)^2,$$

$$u_1=\frac{1}{m^2}(q-k_2)^2,\quad u_2=\frac{1}{m^2}(q-k_3)^2, \tag{3}$$

m — being the pion mass.

One more invariant variable $t_2=m^{-2}(q-k_2-k_3)^2$ is a linear function of the integration variable ζ:

$$t_2=u_1-\frac{1}{2}(1+u_1)(1-u_2)-\frac{1}{2}(1-u_1)(1-u_2)[\xi\eta-\sqrt{(1-\xi^2)(1-\eta^2)}\zeta\ ], \tag{4}$$

where

$$\xi=\frac{\lambda(1,s_2,s_2')-(1-s_1)^2+(1-u_1)^2}{2(1-u_1)\sqrt{\lambda(1,s_2,s_2')}},\quad \eta=\frac{(1-s_3')^2-(1-u_2)^2-\lambda(1,s_2,s_2')}{2(1-u_2)\sqrt{\lambda(1,s_2,s_2')}}, \tag{5}$$

$\lambda(x,y,z)=x^2+y^2+z^2-2(xy+xz+yz)$ is a conventional triangle function, and

$$s_2'=1+s_2-u_1-s_1,\quad s_3'=2-s_1-u_1-u_2. \tag{6}$$

At last, the limits of integration for the variable u_2 in (2) are

$$u_2^{\pm}=1-\frac{1}{2}(u_1+s_1)\pm\frac{1}{2}\sqrt{\lambda(1,s_2,s_2'}. \tag{7}$$

The function F stands for a half sum of the squared helicity amplitudes:

$$F=\sum_{\{\lambda\}}|M_{+\lambda_2\lambda_3\lambda_4}|^2. \tag{8}$$

To evaluate these helicity amplitudes, it is convenient to use the light–light scattering tensor from Ref. 10. In fact, as the virtual mass of the intermediate photon is $\sim m$ and much bigger than the electron mass, we have used the asymptotic form of the light–light scattering amplitudes for a massless electron in the loop.[11,12]

Let us note, however, that we cannot use the polarization vectors from Ref. 10 because of an additional photon and the need for photon permutations. Instead we have taken polarization vectors in a form which has proven useful in various QED calculations:[13]

$$\varepsilon_\mu^{(\lambda m)}(k)=N_m[q^{(m)}\cdot k\ p_\mu^{(m)}-p^{(m)}\cdot k\ q_\mu^{(m)}+i\lambda_m\varepsilon_{\mu\nu\lambda\sigma}p^{(m)\nu}\ k^\lambda\ q^{(m)\sigma}],$$
$$N_m^{-1}=2\sqrt{p^{(m)}\cdot q^{(m)}p^{(m)}\cdot k\ q^{(m)}\cdot k},\quad m=1-4,\quad p^{(m)^2}=q^{(m)^2}=0. \tag{9}$$

For the different photons here we take

$$p^{(1)}=k_2,\quad p^{(2)}=k_3,\quad p^{(3)}=k_4,\quad p^{(4)}=k_1,$$
$$q^{(1)}=k_4,\quad q^{(2)}=k_1,\quad q^{(3)}=k_2,\quad q^{(4)}=k_3. \tag{10}$$

The polarization vectors from Ref. 10 can also be expressed in this form ($8\Delta=k_2\cdot k_3\ k_3\cdot k_4\ k_2\cdot k_4$):

$$u_\mu^{\prime(-\lambda_2)}=\frac{1}{4\sqrt{2\Delta}}[k_4\cdot k_2\ k_{3\mu}-k_3\cdot k_2\ k_{4\mu}+i\lambda_2\varepsilon_{\mu\nu\lambda\sigma}k_3^\nu k_2^\lambda k_4^\sigma],$$

$$u_\mu^{(\lambda_3)}=\frac{1}{4\sqrt{2\Delta}}[k_2\cdot k_3\ k_{4\mu}-k_4\cdot k_3\ k_{2\mu}+i\lambda_3\varepsilon_{\mu\nu\lambda\sigma}k_4^\nu k_3^\lambda k_2^\sigma]\equiv\varepsilon_\mu^{(\lambda_3)}, \tag{11}$$

$$u_\mu^{(-\lambda_4)}=\frac{1}{4\sqrt{2\Delta}}[k_3\cdot k_4\ k_{2\mu}-k_2\cdot k_4\ k_{3\mu}+i\lambda_4\varepsilon_{\mu\nu\lambda\sigma}k_2^\nu k_4^\lambda k_3^\sigma].$$

But $u_\mu^{\prime(-\lambda_2)}$ and $u_\mu^{(-\lambda_4)}$ differ by phase factors from $\varepsilon_\mu^{(\lambda_2)}$ and $\varepsilon_\mu^{(\lambda_4)}$ (note that, in contrast with Ref. 10, $u_\mu^{\prime(-\lambda_2)}$ corresponds to the $+\lambda_2$ circular polarization for the second photon, because now it too is outgoing). Therefore, when using the expressions from Ref. 10 we should not forget the relevant phase factors. For example,

$$u^{\prime(\lambda_2)}\cdot\varepsilon^{(\lambda_2)}=\frac{N_2\ k_2\cdot k_3}{2\sqrt{2\Delta}}\ \Phi(\lambda_2;1234),$$

where

$$\Phi(\lambda;1234)=k_1\cdot k_2\ k_3\cdot k_4+k_1\cdot k_3\ k_2\cdot k_4-k_1\cdot k_4\ k_2\cdot k_3$$
$$+i\lambda[k_1,k_2,k_3,k_4],\quad [k_1,k_2,k_3,k_4]=\varepsilon_{\mu\nu\lambda\sigma}k_1^\mu k_2^\nu k_3^\lambda k_4^\sigma. \tag{12}$$

Owing to the remarkable cyclic symmetry in the definition (9), (10) of the polarization factors $\varepsilon_\mu^{(\lambda_i)}$ for the $\pi^0\rightarrow 4\gamma$ helicity amplitudes we get [remember that some numerical factors have already been taken out in (1)]:

$$M_{\lambda_1\lambda_2\lambda_3\lambda_4}=\frac{1}{k_1\cdot k_2\ k_1\cdot k_3\ k_1\cdot k_4\ k_2\cdot k_3\ k_2\cdot k_4\ k_3\cdot k_4}$$

$$\times\sum_{\text{cyclic}}\frac{\Phi(\lambda_2;1234)\Phi(\lambda_4;1432)}{k_2\cdot k_4}\{A(\lambda_1;1234)\varepsilon^{(1)}_{-\lambda_2,\lambda_3,\lambda_4}(234)$$

$$+B(\lambda_1;1234)\varepsilon^{(1)}_{-\lambda_2,\lambda_4,\lambda_3}(243)+2C(\lambda_1;1234)\varepsilon^{(2)}_{-\lambda_2,\lambda_3,\lambda_4}(234)\}. \quad (13)$$

Here the summation extends over simultaneous cyclic permutations of $(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ and (k_1,k_2,k_3,k_4):

$$\sum_{\text{cyclic}}F(1234)=F(1234)+F(2341)+F(3412)+F(4123).$$

The $\varepsilon^{(1)}_{\{\lambda\}}$ and $\varepsilon^{(2)}_{\{\lambda\}}$ amplitudes have been defined in Ref. 11 (see also Ref. 12). For the functions A, B, and C we have

$$A(\lambda_1;1234)=\varepsilon^{\mu\nu\lambda\sigma}k^1_\mu\ q_\nu\ \varepsilon^{(\lambda_1)}_\lambda\left[k_3-\frac{(q-k_1)\cdot k_3}{(q-k_1)\cdot k_2}k_2\right]_\sigma,$$

$$B(\lambda_1;1234)=-\varepsilon^{\mu\nu\lambda\sigma}k^1_\mu\ q_\nu\ \varepsilon^{(\lambda_1)}_\lambda\left[k_4-\frac{(q-k_1)\cdot k_4}{(q-k_1)\cdot k_2}k_2\right]_\sigma, \quad (14)$$

$$C(\lambda_1;1234)=i\ \varepsilon^{\mu\nu\lambda\sigma}\ k^1_\mu\ q_\nu\ \varepsilon^{(\lambda_1)}_\lambda\ \varepsilon_{\sigma\mu'\nu'\lambda'}\ (k_1-q)^{\mu'}\ k_2^{\nu'}\ k_3^{\lambda'}.$$

The polarization vectors (9) and (11) become ill-defined for collinear photons. Fortunately, this kinematical region gives a negligible contribution to the decay width. In fact, the corresponding fictitious kinematical singularities don't cause any considerable trouble in numerical calculations because the phase factors also vanish for collinear photons, and only the singularities corresponding to the three simultaneously collinear photons remain.

There are no infrared divergences in our problem (when the energy of any photon goes to zero), as is easily seen from the explicit expressions of the $\varepsilon^{(1)}$, $\varepsilon^{(2)}$ amplitudes. The contribution of the muon in the fermion loop can be neglected at least to order $(\omega/m_\mu)^3$, $\omega\sim m/4$, where ω is the mean photon energy, due to the known low energy behavior of the light-light scattering amplitude.

The numerical calculations give the result

$$Br(\pi^0\to4\gamma)=(2.6\pm0.1)\cdot10^{-11}. \quad (15)$$

This is about three orders of magnitude below the present experimental limit.

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