Kinetics of avalanche mixing of granular materials

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The problem of avalanche mixing of two fractions of a granular material is solved. The fractions are mixed in a cylinder that rotates slowly around its horizontally oriented longitudinal axis. The cylinder is not completely filled. At each instant of time, the mixing occurs only in the surface layer of granules. The diffusion with convection and rapid mixing is described. © *1995 American Institute of Physics.*

After the appearance of the pioneering papers[1–3] on self-organized criticality, the number of works (primarily experiments and computer simulations) on the different aspects of mixing of granular materials also increased sharply.[4–7] The physics of the mixing process turned out to be very nontrivial, and the patterns arising in experiments are often so striking that the illustration in Ref. 7 made the cover of the March issue of *Nature*. This illustration motivated the publication of this letter.

1. We shall consider a very simple model in which the mixing process can be described analytically. Two types of granules — black and white — are poured into a cylinder whose longitudinal axis is oriented horizontally; otherwise, the granules of the different fractions are indistinguishable. The cylinder is in a gravitational field and is not completely filled. Initially, the black fraction is poured on top of white fraction (see Fig. 1a). The cylinder starts to rotate slowly and adiabatically around its longitudinal axis (counterclockwise, for definiteness). Therefore, the angle of rotation of the cylinder plays the role of time. We assume that the problem is uniform along the axis of the cylinder and we ignore the longitudinal displacements of the granules.[5,6] We can therefore talk about areas instead of volumes. The answer does not depend on the radius of the cylinder, which can be assumed equal to 1. We shall characterize the volume of the unfilled space and the black fraction by the aperture angles θ and χ, respectively (see Fig. 1a). We shall describe the state of each point of the material by the quantity ρ — the fraction of the black material at a given location ($\rho=1$ at the location where all of the material is black, and $\rho=0$ at the location where all of the material is white). We set the total density of the granular material equal to unity, so that $\rho(\mathbf{x},t)$ plays the role of the density of the black fraction at the point $\mathbf{x}$ at the time t.

We assume that the granules cannot intersperse and slip relative to the surface of the cylinder as long as they are located in the bulk of the material. They intersperse only when they emerge on the free surface (see Fig. 1b). As the cylinder rotates, avalanches, in which the fractions can mix, continuously fall from the free surface. Such mixing is called avalanche mixing.[7] In a real experiment the surface is flat and when the cylinder rotates slowly, the surface is tilted at the friction angle (strictly speaking, this angle

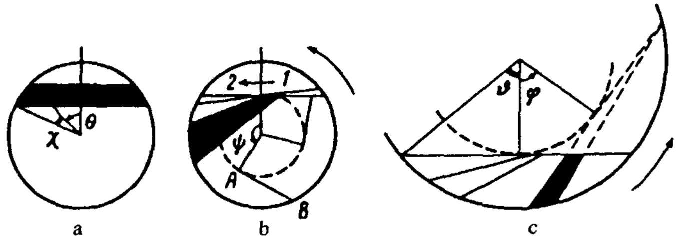

FIG. 1. a — Arrangement of the fractions before the cylinder starts to rotate. b — Interspersion scheme at one instant in time. For an infinitely small rotation of the cylinder, the granules of the different fractions are transferred from sector *1* into sector *2*, mixing uniformly. The density of the black material along any tangent of the type AB is therefore the same after one revolution. The angle ϕ fixes the position of each such tangent. c — One instant of time $0<t<2\vartheta$ ($\vartheta\equiv\pi-\theta$) in the case where the cylinder is less than half filled.

fluctuates slightly with time, but these variations are small[7] and we shall disregard them). The friction angle does not appear in our answers, so that we can set it equal to zero.

We must make one more strong assumption. We assume that the material in the avalanches is completely mixed, i.e., to a homogeneous state (see Fig. 1b). It is natural to choose this assumption as an initial approximation, if subtleties associated with the structure of the granules (sticking, hooking, and so on) are ignored. As the cylinder rotates, ρ will then be the same at all points on the left half of the free surface. After the first revolution we can therefore introduce $\rho(\psi,t)$ — the density of the black fraction at the time t at the points of the tangent of the type designated as AB in Fig. 1b (ψ is the angle between the corresponding radius vector and the normal to the free surface) — the main quantity which we will use below.

Computer simulation[7] of a model such as the one considered here showed that it describes a real experiment surprisingly well.

One can see immediately that for $\theta<\pi/2$ (the drum is more than half-filled) mixing occurs only in the ring $\cos\theta<r<1$. The central part ($r<\cos\theta$) simply rotates together with the cylinder. For $\pi/2<\theta<\pi$ (the cylinder is less than half-filled) the material is completely mixed.

Let us first consider after what period of time T is $\rho>0$ everywhere except in the central region in the case $\theta<\pi/2$, i.e., when the maximum possible volume of the white material is first "dirtied"? For simplicity, let the fraction of the black material be small: $\chi\to 0$. It is obvious that for $\pi/2<\theta<\pi$ we have $T=2(\pi-\theta)$. If $\theta<\pi/2$, however, then after each revolution the total "angle" of the purely white fraction in the outer ring decreases by 2θ. As can easily be verified, T will therefore be equal to the sum of $2\pi[2\pi/(2\theta)]$ and $2\pi-2\theta[2\pi/(2\theta)]$; here the brackets denote the integer part. We then have

$$\frac{2T}{2\pi}=1+\left(1-\frac{\theta}{\pi}\right)\left[\frac{\pi}{\theta}\right] \tag{1}$$

(see Fig. 2).

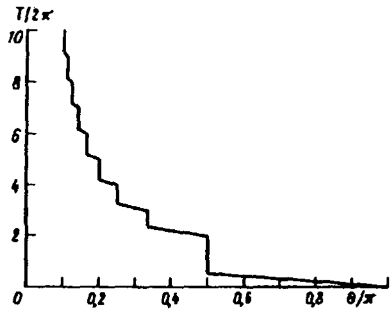

FIG. 2. Time at which at all points, except in the central region, in the case $\theta < \pi/2$ for the first time $\rho > 0$. The limiting curve for $\chi \to 0$ (the amount of black material is small) is displayed.

2. We shall now describe the kinetics of avalanche mixing in the case where the cylinder is more than half-filled — $\theta < \pi/2$. After the first revolution, however, the state of the system can be described, as we have indicated, by the distribution $\rho(\psi,t)$. The total amount of the black fraction beyond the radius $1 \cdot \cos\theta$ will then be

$$M = \frac{\sin^2\theta}{2} \int_0^{2\pi-2\theta} d\psi \rho(\psi,t) + \frac{\cos^2\theta}{2} \int_{2\pi-2\theta}^{2\pi} d\psi \rho(\psi,t) \tan^2\left(\frac{2\pi-\psi}{2}\right). \tag{2}$$

Here the coefficient of the first integral is present because of the integration along the tangent $\int_0^{\sin\theta} dx\, x$ in the region of angles ψ of the radius vector in the interval $(0,2\pi-2\theta)$. When these angles fall between $2\pi-2\theta$ and 2π, however, the tangent is cut by the free surface, the length of the tangent, as is easily verified, becomes equal to $\cos\theta \tan[(2\pi-\psi)/2]$, and we obtain the second integral in Eq. (2). Since mixing occurs only at the free surface, for $0 < \varphi < 2\pi$ and $t > \varphi$ we have $\rho(\varphi,t) = \rho(0, t-\varphi)$. Substituting this relation into Eq. (2) and differentiating with respect to time, we obtain an equation that shows how $\rho(\varphi)$ changes in one revolution:

$$\rho(\varphi,t+2\pi) = \cot^2\theta \int_0^{2\theta} d\zeta \rho(\varphi-\zeta,t) \frac{\sin(\zeta/2)}{\cos^3(\zeta/2)} \tag{3}$$

— a discrete linear mapping. In a simple case $\theta \ll \pi/2$ the right-hand side can be expanded in powers of θ and, introducing for convenience a "discrete" time $\tilde{t} = 2\pi[t/(2\pi)]$ [we are describing a slow change of $\rho(\varphi)$, observing this distribution at the times separated from one another by integral periods], we obtain the equation

$$2\pi \frac{\partial \rho}{\partial \tilde{t}} = -\frac{4}{3}\theta \frac{\partial \rho}{\partial \varphi} + \theta^2 \frac{\partial^2 \rho}{\partial \varphi^2}. \tag{4}$$

This equation is valid for $\tilde{t} \gg 2\pi$. In the case at hand, avalanche mixing thus reduces to a diffusion process with convection. If it is assumed, for simplicity, that there is only a small amount of the black material ($\chi \ll \theta$), then at first all of the black material will be concentrated at small angles, and the initial condition can be assumed to be $\rho(\varphi,0) = 4\chi\delta(\varphi)$. We will then obtain

$$\rho(\varphi,\tilde{t})=4\chi\left\{\frac{1}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\exp\left(-n^2\frac{\theta^2}{2\pi}\tilde{t}\right)\cos\left[n\left(\varphi-\frac{2}{3\pi}\theta\tilde{t}\right)\right]\right\}$$

$$=\frac{4\chi}{2\pi}\theta_0\left(\frac{\varphi-2\theta\tilde{t}/(3\pi)}{2\pi},\frac{\theta^2}{2\pi^3}\tilde{t}\right), \tag{5}$$

where θ_0 is the theta function.[8] At long times $\tilde{t}\gg\pi^3/\theta^2$ exponential relaxation occurs with the characteristic time $t_e=2\pi/\theta^2$ to the uniform distribution $\rho_\infty=2\chi\theta^2/(\pi\theta^2)=2\chi/\pi$. At shorter times, however,

$$\rho(\varphi,\tilde{t})=\frac{4\chi}{\theta\sqrt{2\tilde{t}}}\exp\left\{-\frac{\pi}{2\theta^2\tilde{t}}[\varphi-2\theta\tilde{t}/(3\pi)]^2\right\}, \tag{6}$$

which is the standard result for linear diffusion with convection on an infinite interval, where the center of the distribution drifts with the velocity $2\theta/(3\pi)$.

3. Let us now discuss the case $\pi/2<\theta<\pi$ — a less than half-full cylinder (see Fig. 1c), in which mixing occurs much more rapidly in the entire volume. Here, in contrast to Sec. 2, the tangents along which $\rho=$ const always intersect the free surface. It is more convenient to use the quantity $\rho(0,t)$ — the fraction of the black material at the points of the tangent which at the time t coincides with the free surface — to describe the development of the mixing process over time. Let $\vartheta\equiv\pi-\theta$. In the case $t>2\vartheta$ (i.e., when the initial stage of mixing has been completed), proceeding in the same manner as in the previously analyzed case, we obtain the equation

$$\rho(0,t)=\cot^2\vartheta\int_0^{2\vartheta}d\zeta\rho(\zeta,t)\frac{\sin(\zeta/2)}{\cos^3(\zeta/2)}. \tag{7}$$

If $0<t<2\vartheta$, however, then an additional term which accounts for the dropping of granules of the "purely black" fraction must be introduced on the right-hand side of the equation:

$$\rho(0,t)=\cot^2\vartheta\int_0^{t}d\zeta\rho(\zeta,t)\frac{\sin(\zeta/2)}{\cos^3(\zeta/2)}+\frac{2}{\sin^2\vartheta}\frac{\partial S}{\partial t}(t), \tag{8}$$

where $\partial S(\varphi)/\partial\varphi$ is the derivative of the area of the dashed triangle in Fig. 1c and which characterizes the additional contribution from the black fraction. In the simplest case $\chi\ll\vartheta\ll\pi/2$ (i.e., when the black fraction is much smaller than the white fraction), we can obtain by means of basic geometry the following expression for this derivative:

$$\partial S(\varphi)/\partial\varphi=\tfrac{1}{2}\tfrac{4}{3}\chi(\vartheta+\varphi/2).$$

[In fact, this expression must also contain a nonlinear part (a peak) at small angles $\varphi<\chi$, which corresponds to a rapid release of the black fraction in the avalanches just as the mixing begins. As $\chi\to0$, it can be cut out by renormalizing χ beforehand — by introducing an additional factor 4/3 in order to preserve the correct total amount of the black fraction.] As a result, introducing the "dimensionless" quantities $\tau\equiv t/(2\vartheta)$ and $\nu(\tau)\equiv\rho(0,t)/(4\chi/3\vartheta)$, we obtain for $\vartheta\ll\pi/2$ the expressions

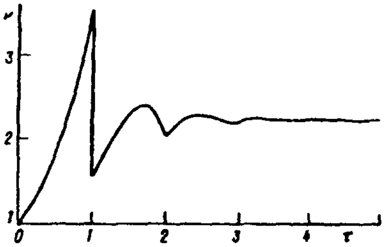

FIG. 3. Density of the black fraction on the left half of the free surface (see Fig. 1c) as a function of time in the case where the total volume of material is small compared to the volume of the empty space and the volume of the black fraction is small compared to the volume of the white fraction: $\tau \equiv t/(2\vartheta)$, $\nu(\tau) \equiv \rho(0,t)/[4\chi/(3\vartheta)]$.

$$\nu(\tau)=2\int_0^\tau d\epsilon(\tau-\epsilon)\nu(\epsilon)+1+\tau, \quad 0<\tau<1, \quad \nu(\tau)=2\int_{\tau-1}^\tau d\epsilon(\tau-\epsilon)\nu(\epsilon), \quad \tau>1. \tag{9}$$

Here the solution of the first equation is used as the initial condition for the second equation. Differentiating twice, we obtain

$$\nu''(\tau)=2\nu(\tau), \quad 0<\tau<1, \quad \nu''(\tau)=2\nu(\tau)-2\nu(\tau-1)-2\nu'(\tau-1), \quad \tau>1, \tag{10}$$

where $(\cdot)' \equiv d/d\tau$, and at the initial instant of time $\nu(0)=\nu'(0)=1$. The solution of the first of these equations has the form

$$\nu(\tau)=\frac{1}{2\sqrt{2}}[(\sqrt{2}+1)\exp(\sqrt{2}\tau)+(\sqrt{2}-1)\exp(-\sqrt{2}\tau)], \tag{11}$$

so that $\nu(1-)=3.546$ and $\nu'(1-)=4.915$. The second equations, however, from Eq. (9) or (10) can be solved by using Laplace transformation. The final result is

$$\nu(\tau)=\frac{9}{4}+\sum_j \frac{2(z_j+1)}{z_j^2(z_j+2)}e^{z_j(\tau-1)}\int_0^1 d\epsilon\,\nu(\epsilon)[1+z_j(1-\epsilon)-(1+z_j)e^{-z_j\epsilon}], \tag{12}$$

where the summation extends over all roots (except zero roots) of the characteristic equation $z^2-2+2(z+1)\exp(-z)=0$ (Ref. 9). These roots are arranged as follows. There is a triply degenerate zero root, which leads to the appearance of the term 9/4 — it corresponds to uniform mixing with density $\rho_\infty = 3\chi/\vartheta$ at long times. There is also a collection of complex conjugate roots with asymptotic values $z_{\pm p}=-\log(p\pi)\pm(2p-3/2)\pi i+O(\log p/p)$, where p is a positive integer, greater than 1. For all $p \geq 2$, these values can be easily refined by repeatedly iterating the relation $z=\log[2(z+1)/(2-z^2)]$. Performing the summation in Eq. (12), we obtain the function shown in Fig. 3. Oscillations such as those which we obtained have been observed experimentally.[7]

At $\tau \approx 2.5$, the function $\nu(\tau)$ is described very well by taking into account the two lowest $(p=2)$ roots of the characteristic equation. Inserting the numerical values of the coefficients, we obtain the very compact expression

$$\nu(\tau)=9/4+0.946\exp[-1.392(\tau-1)]\cos[7.553(\tau-1)-1.336]. \qquad (13)$$

We now call attention to the fact that the oscillation period of the cosine is less than 1. In the region $1<\tau<2$ the result can be put into the following analytic form without any special difficulty:

$$\nu(\tau)=[\nu(1-)-2-3(\tau-1)/2]\cosh[\sqrt{2}(\tau-1)]+[(\nu'(1-)$$
$$-3/2)/\sqrt{2}-\sqrt{2}(\tau-1)]\sinh[\sqrt{2}(\tau-1)]. \qquad (14)$$

In conclusion, we note that the "phase" of the granules that slip along the free surface changes abruptly. Here, the surface thus plays the role of a "phase-slip center" — a concept which is encountered in the most diverse fields of solid state physics (see, for example, Refs. 10–12).

In summary, we have shown that two mixing regimes arise in our problem: a diffusion with convection regime with a more than half-filled cylinder and a rapid mixing regime with a less than half-filled cylinder. It was found that although the problem has a nontrivial solution, it avoids dealing with complicated nonlinear mappings.[13]

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