Linear transformation of Rossby waves in shear flows

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The dynamics of a shallow, nonuniformly rotating atmosphere in the presence of a zonal shear flow is studied. A new effective mechanism, which is ignored in the standard quasigeostrophic approximation that leads to the Charney–Obukhov equation, is demonstrated for the transformation of Rossby waves into inertial waves. The discovery of this phenomenon was made possible by the application of the currently highly popular nonmodal analysis, in hydrodynamics, of the evolution of disturbances in hydrodynamic shear flows. © *1995 American Institute of Physics.*

Terrestrial and astrophysical flows are a result of the combination of many physical phenomena, each of which is manifested differently in the formation and evolution of the wave processes that are characteristic of these flows. In the mathematical description and physical interpretation of wave processes it is assumed at the outset that the physical phenomena are separated according to their importance in each specific case and the secondary phenomena are "sifted out.." However, the large number of physical phenomena—potential participants in the wave motions—makes this sifting a subtle process which is fraught with the danger that some of the factors will be underestimated. Ultimately, this problem gives us at least an incomplete and sometimes a substantially distorted picture of the wave evolution.

In the present paper we wish to discuss the miscalculation that is made in describing the evolution of Rossby waves in the presence of zonal shear flows. Specifically, we shall demonstrate that in flows with moderately strong shear the low-frequency Rossby waves, which are predominantly rotational, transform in time into high-frequency waves — potential inertial waves. Actually, we are talking here about a substantial change in the time scale of the wave process as a result of the transformation. This new type of transformation of waves which occurs in shear flows was first described in Ref. 1 for the case of MHD waves. The physics of the process is simple and can be easily understood using the example of a system of coupled linear oscillators. Consider two pendulums whose lengths are time-dependent. This causes the characteristic frequencies of these pendulums to be time-dependent: $w_1(t)$ and $w_2(t)$. We assume that the pendulums is weakly coupled. Let $\chi(t)$ be the coupling constant (which, in general, is also time-dependent). The oscillations of such coupled pendulums can then be written in the form

$$\ddot{x}_1 + w_1^2(t)x_1 + \chi(t)x_2 = 0, \tag{1}$$

$$\ddot{x}_2 + w_2^2(t)x_2 + \chi(t)x_1 = 0. \tag{2}$$

If the frequencies of these pendulums differ strongly from one another, then despite the coupling, there is virtually no exchange of energy between them. Effective energy exchange starts when the frequencies of the oscillators come close to one another. The necessary conditions for effective exchange of energy are:[2]

(**A**) existence of a "region of degeneracy" in which $|w_1^2(t) - w_2^2(t)| \leq |\chi(t)|$;

(**B**) "slow" traversal of the "region of degeneracy" —over a period of time that is much longer than $\chi(t)$: $|\dot{w}_1(t)|, |\dot{w}_2(t)| \ll |\chi(t)|$.

If at first only the first pendulum oscillates, then because of the change in its length, the frequencies $w_1(t)$ and $w_2(t)$ can come close to one another and the conditions (**A**) and (**B**) will be satisfied during a certain limited interval of time. A large (and possibly the main) part of the vibrational energy of the first pendulum is transferred to the second pendulum. As a result, strong oscillations of the second pendulum start. In the process, the first pendulum may stop completely. We are describing this process in detail, because we shall discuss a similar scenario for Rossby waves.

The conditions (**A**) and (**B**) are valid for arbitrary, coupled, oscillatory systems, to which the description of an entire series of natural physical processes can be reduced. They are also directly applicable to the analysis of the linear interaction of waves on different branches as the frequencies of the waves approach one another. There arises the question: Under what circumstances are the frequencies of the propagating waves time-dependent? To answer this question, a nonmodal approach, widely used in the last few years (see, for example, Refs. 3–5), must be used to describe the evolution of disturbances in hydrodynamic shear flows. For an entire series of shear flows (planar Couette flow, Poiseuille flow, and circular Poiseuille flow) the linearized operator arising in the problem of linear stability is found to be substantially *nonorthogonal*; i.e., the eigenfunctions (modes) are far from being mutually orthogonal. This leads to strong interference between the characteristic modes. As a result, even if all characteristic modes decay monotonically with time, a particular solution of the problem can reveal large relative growth over a finite time interval. For this reason, analysis of a separate characteristic mode, performed on the basis of the standard modal approach in which interference is disregarded, can lead to a number of errors.

Consider a two-dimensional shear flow $\mathbf{V}_0 = (Sy, 0)$, where S is the shear parameter. It is well known that because of the existence of shear, the propagating disturbances are not simply plane waves, since the wave crests are rotated by the nonuniform average flow.[5] The wave vector is found to depend on the time: If a Fourier harmonic with the wave numbers k_x and $k_y(0)$ is excited initially $[\varphi(0) \sim \exp(ik_x x + ik_y(0)y)]$, then with time the wave number along the Y axis will change according to the law

$$k_y(t) = k_y(0) - Stk_x, \tag{3}$$

which, in turn, gives rise to a time-dependence of the frequency characteristic of the spatial Fourier harmonics $w = w(t)$. In other words,

$$\varphi(t) \sim \exp\left[-i \int_0^t w(t')dt' + ik_x x + i(k_y(0) - Stk_x)y \right]. \quad (4)$$

In this case, if at least two wave branches exist in the flow, then we should first investigate the possibility that the conditions (**A**) and (**B**) are satisfied and then the possible mutual transformation of the waves.

The defect that harbors the standard approximations employed for describing Rossby waves can be formulated as follows in the context of what we have said above. The characteristic oscillation time of these waves is much longer than the period of revolution of the planet. In investigating these waves only the terms which are due to the slowness of the process are therefore retained in the equations. Of course, this approximation precludes fast processes in the system and ignores the possible transformation of Rossby waves into high-frequency gyroscopic waves in the presence of zonal shear flows. It can thus strongly distort the picture of wave processes in the atmosphere.

We shall now analyze in detail the transformation of Rossby waves into waves on the high-frequency branch. Barotropic modes in the atmosphere and in the oceans are described by the system of shallow-water equations with the addition of the Coriolis force:[6]

$$\partial_t \mathbf{U} + (\mathbf{U}\cdot\nabla)\mathbf{U} + 2\Omega \sin \phi[\mathbf{e}\times\mathbf{U}] + \nabla P = 0, \quad (5)$$

$$\partial_t H + \nabla\cdot(H\mathbf{U}) = 0. \quad (6)$$

Here H is the thickness of the liquid layer, $\mathbf{U}=(U_x, U_y)$ is the horizontal velocity in a local tangential plane, $\mathbf{e}$ is a vertical unit vector, Ω is the rotational frequency of the planet, and ϕ is the latitude. The quantity P is the anomaly of the geopotential and is proportional to the deviation of the free surface from the equilibrium state as determined from the condition of hydrostatic equilibrium

$$P = g(H - H_0), \quad (7)$$

where H_0 is the unperturbed thickness of the layer which is assumed to be constant for simplicity, and g is the acceleration of gravity at the surface of the planet.

We assume that the spatial scales of the flows are much smaller than the radius of the planet. This makes it possible to introduce a local Cartesian coordinate system (x,y) with the x axis oriented along the latitude and the y axis oriented along the meridian. In this case we can use the linear approximation for the Coriolis parameter—the β-plane approximation: $2\Omega \sin \phi = f_0 + \beta y$, where $f_0 = 2\Omega \sin \phi_0$, $\beta = 2\Omega(\cos \phi_0/a)$, and a is the radius of the planet. In the further analysis the small parameters in the asymptotic expansion are, as a rule, the quantity $\epsilon = (T_0 f_0)^{-1}$ and the Rossby number $R = V_0/Lf_0$, where T_0 and L are the characteristic temporal and spatial scales of the disturbances. In the lowest order in the expansion in the Rossby number, which actually corresponds to averaging over the high-frequency inertial branch of the oscillations, we obtain the Charney–Obukhov equation.[7] Most studies of the dynamics of Rossby waves involve the analysis of this equation.[6–8] We shall show below that when the Charney–Obukhov equation is used as a base model, important energy exchange processes between low-frequency waves (Rossby waves) and the high-frequency waves (inertial waves) are ignored, even in the case of a simple shear flow.

It is easy to see that in the presence of a constant shear the Rossby number can be defined as $R=(\partial V_0/\partial x)/f_0$. The dimensionless parameter characterizing the ratio of the time scales will be not the Rossby number but rather the new parameter $R^*=R|1-R|^{-1/2}$, which is not small, even for moderate values of R. This narrows substantially the range of application of the Charney–Obukhov model in the analysis of the effect of shear flows on the evolution of Rossby waves.

To reveal the details of what happens, we shall study the complete system of equations for a shallow atmosphere and we shall linearize them with respect to a planar zonal shear flow. Let the velocity field and thickness of the liquid layer be $\mathbf{U}=\mathbf{V}_0+\mathbf{V}$ and $H=H_0+h$, where $\mathbf{V}_0=(Sy,0)$ and H_0 are the steady-state solution of the system (5), and (6). After linearization for small $\mathbf{V}$ and h and separation of the velocity field into rotational and irrotational components, we obtain the following system of equations for the vorticity $\omega=\mathrm{curl}\mathbf{V}=\partial_x V_y-\partial_y V_x$, the divergence $\xi=\nabla\cdot\mathbf{V}=\partial_x V_x+\partial_y V_y$, and the deviation h of the layer thickness:

$$\partial_t\omega+Sy\partial_x\omega-S\xi+(f_0+\beta y)\xi=0, \tag{8}$$

$$\partial_t\xi+Sy\partial_x\xi+2S\partial_x V_y+\beta V_x-(f_0+\beta y)\omega+g\Delta h=0, \tag{9}$$

$$\partial_t h+Sy\partial_x h+H_0\xi=0. \tag{10}$$

Let us consider a solution in the form

$$\begin{pmatrix}\omega\\ \xi\\ h\end{pmatrix}=\begin{pmatrix}\hat{\omega}(t)\\ \hat{\xi}(t)\\ \hat{h}(t)\end{pmatrix}\exp(ik_x x+ik_y(t)y). \tag{11}$$

This approach was employed in Ref. 9 to analyze the effect of shear on the β-effect in an extremely simplified model which ignore the effects of compressibility and, correspondingly, inertial waves.

We switch to the dimensionless variables

$$\tau=f_0 t,\quad k_x\Rightarrow\frac{k_x}{L},\quad k_y\Rightarrow\frac{k_y}{L},\quad \mathbf{V}\Rightarrow\mathbf{V}V_0,\quad h\Rightarrow h\frac{V_0 f_0 L}{g},\quad \beta\Rightarrow\frac{\beta f_0}{L}. \tag{12}$$

We obtain the following equations for the amplitude functions:

$$\left(\partial_\tau-i\frac{\beta k_x}{k^2(\tau)}\right)\hat{\omega}-(1-\varepsilon)\hat{\xi}=0, \tag{13}$$

$$\left(\partial_\tau-i\frac{\beta k_x}{k^2(\tau)}\right)\hat{\xi}+2\varepsilon\frac{k_x k_y(\tau)}{k^2(\tau)}\hat{\xi}+\left(1-2\varepsilon\frac{k_x^2}{k^2(\tau)}\right)\hat{\omega}-k^2(\tau)\hat{h}=0, \tag{14}$$

$$\partial_\tau\hat{h}+\eta\hat{\xi}=0. \tag{15}$$

Here

$$\varepsilon=\frac{S}{f_0},\quad \eta=\frac{gH_0}{f_0^2L^2}=\frac{Ro^2}{L^2},\quad \text{and } Ro=\frac{(gH_0)^{1/2}}{f_0}$$

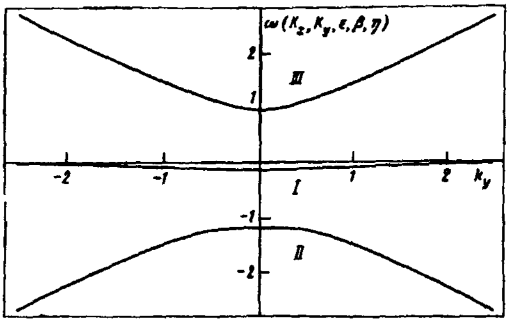

FIG. 1. Dispersion curves for $\varepsilon=0$, $\beta=0.05$, $\eta=1.0$, $k_x=0.4$.

is the Rossby radius. In deriving Eqs. (13)–(15) we assumed that the rotational nonuniformity βy is small compared to the frequency f_0 in the case where these terms operate on the same variable [i.e., $(f_0+\beta y)F \approx f_0 F$]. This approximation corresponds to the case in which the transverse scale of the disturbance is small compared to the earth's radius, and it is identical to the applicability of the β-plane approximation. The energy of the spatial Fourier harmonic (13)–(15) has the form

$$E_k=\frac{(\hat{\omega}\hat{\omega}^*+\hat{\xi}\hat{\xi}^*)}{k^2(\tau)}+\frac{\hat{h}\hat{h}^*}{\eta}. \tag{16}$$

In graphical representations of the dispersion curves of inertial and Rossby waves, one generally uses the dependence on the latitudinal wave vector k_x. In our case, however, to visualize more clearly the wave-transformation phenomenon under discussion, it is more convenient to study the dependence of the frequency on k_y. In the shear flow we are examining k_y depends on the time. Taking this fact into account, in our plots it is easier to follow the change in the frequency characteristic of the waves in time.

It should be noted that for $\varepsilon \neq 0$, because the wave amplitudes are time-dependent, the dispersion equation that can be derived from Eqs. (13)–(15) is, strictly speaking, highly conditional. Nonetheless, it makes it possible to understand qualitatively correctly the change in time of the frequency characteristic of the waves and also to estimate the degree of convergence of the different wave branches which occurs for certain values of $k_y(\tau)$.

Figure 1 shows the solutions of the dispersion equation for $\varepsilon=0$. In this case there are three wave branches (see curves *I*, *II*, and *III*). The branch *I*, describing Rossby waves, practically merges with the k_y axis, since the frequency of these waves is much lower than that of the other two waves (*II* and *III*). It is obvious that the conditions (**A**) and (**B**) are far from being satisfied. Therefore, Rossby waves are not coupled with inertial waves, and for this reason at $\varepsilon=0$ the mutual transformation of the waves need not be considered.

It is interesting to follow the transformation of the dispersion curves for $\varepsilon \neq 0$ (see Fig. 2). We shall discuss the two branches *I* and *II*, since only the group velocities of these branches can be equal to one another and a resonance coupling can arise only

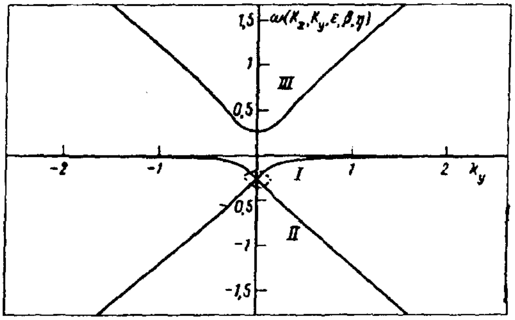

FIG. 2. Dispersion curves for $\varepsilon=0.6$, $\beta=0.05$, $\eta=1.0$, $k_x=0.4$.

between them. Therefore, mutual transformations of only these waves can be expected. Even for shears below the critical value $\varepsilon=0.6$ (for $\varepsilon=1$ the flow becomes unstable[10]) there exists a range of wave numbers k_x, $k_y(\tau)$ in which the low-frequency branch I and the high-frequency branch II converge toward one another.

There arises a "region of degeneracy" (the region represented by a dashed line in Fig. 2) where it is obvious that the conditions for transformation of the waves [conditions (**A**) and (**B**)] are satisfied. In other words, if we assume that initially only a low-frequency Rossby wave with a large value of the wave vector $k_y(0)$ [i.e., $k_y(0)/k_x \gg 1$] is excited, then with time, as $k_y(\tau)$ changes, the frequency of the wave will increase and fall into the region of degeneracy, and a certain part of the energy of the wave will be transformed into the energy of the other branch of waves. This situation is completely analogous to the variable-length interacting pendulums discussed at the beginning of this paper. This qualitative analysis is confirmed completely by the results of a numerical analysis of Eqs. (13)–(15), which is shown, inpart, in Figs. 3 and 4. In our calculations we actually start from a purely low-frequency Rossby wave, whose wave number satisfied the condition

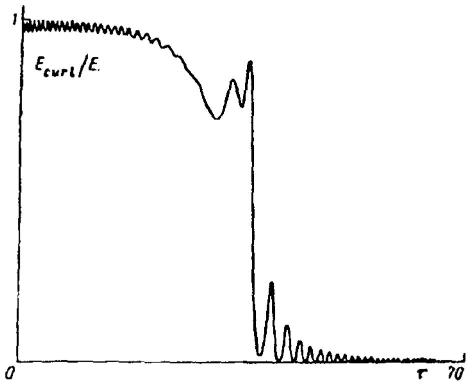

FIG. 3. Ratio of the rotational part of the energy $\hat{\omega}\hat{\omega}^*/k^2(\tau)$ to the total energy E_k; $\varepsilon=0.6$, $\beta=0.05$, $\eta=1.0$, $k_y(0)=10$, $k_x=0.5$, $\tau=0\rightarrow70$.

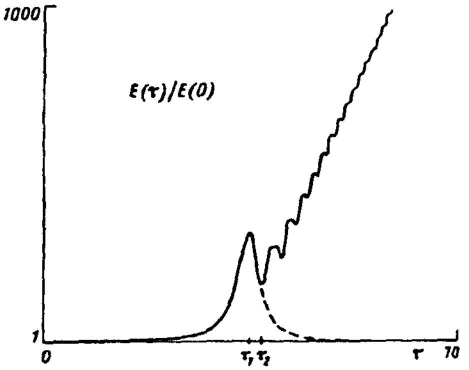

FIG. 4. Total energy of the spatial Fourier harmonic plotted as a function of time: $\varepsilon=0.6$, $\beta=0.05$, $\eta=1.0$, $k_y(0)=10$, $k_x=0.5$.

$k_y(0)/k_x \gg 1$. The fraction of the high-frequency inertial mode in the total energy of the disturbance was initially negligibly small (see the plot at time $\tau=0$ in Fig. 3). For $k_y(0)/k_x \gg 1$ the Rossby waves are predominantly nonpotential waves and the high-frequency inertial waves are predominantly potential waves. We can say, therefore, that the initial energy is concentrated mainly in the nonpotential, part of the disturbance. For $k_y(0)/k_x>0$ and $\varepsilon>0$, $k_y(\tau)$ decreases with time and at $\tau=\tau_1=k_y(0)/k_x$ it drops to zero, then changes sign and increases in absolute value. As one can see from Fig. 3, as the initial disturbance evolves, the fraction of the rotational component in the total energy decreases until it becomes negligibly small; i.e., if for $\tau=0$ the energy was concentrated in the nonpotential, low-frequency modes— Rossby waves, then for $\tau \gg \tau_1$ all of the energy is concentrated in the potential, high-frequency disturbances—inertial waves. The transformation of Rossby waves into inertial waves begins at the time $\tau=\tau_1$, during a limited time interval when the conditions (**A**) and (**B**) are satisfied and these two branches are coupled. It should be noted that the waves of the branches *I* and *II* are not only coupled with one another but they are also coupled with the average flow and they exchange energy with it. This energy exchange is superimposed on the transformation of Rossby waves into high-frequency waves. All of these processes are clearly illustrated in Fig. 4. For $\tau \ll \tau_1$ the transformation conditions are not satisfied and the Rossby waves exchange energy only with the average flow—they draw energy from the shear and grow. The energy of the harmonic of the Rossby waves which is being considered increases to the time $\tau=\tau_1$, after which they start to give energy back to the average flow. However, intensive transformation of a Rossby wave into inertial waves of branch *II* starts in parallel with this process. A large fraction of the energy in the Rossby waves is transformed in the process. At time $\tau=\tau_2$ (see Fig. 4) only a branch-*II* wave remains in the flow. The latter wave grows with time, drawing energy from the shear (see the section of the plot for $\tau>\tau_2$ in Fig. 4). This increase in energy is similar to the process described in Ref. 11. Figure 4 clearly shows how much the process of evolution of a Rossby wave changes as a result of the transformation of the wave into a branch-*II* wave: If the latter process did not occur, then the energy of the Rossby wave would have decreased in the manner represented by the dashed line in Fig. 4.

It is well known that the propagation of Rossby waves in the atmosphere is similar to the propagation of ion-drift waves in a nonuniform plasma.[6,8] The initial shallow-atmosphere equations are virtually identical to the equations for the ion component of the plasma ($T_i \ll T_e$). In deriving the equation for drift waves we actually perform averaging over the cyclotron frequency. It is easy to see that in the presence of a constant-velocity shear the validity of separating the time scales becomes questionable and here, by analogy with Rossby waves, linear transformation of the drift mode into a cyclotron mode and vice versa is possible.

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