Nonlinear conductivity of a disordered medium at the percolation threshold

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It is shown that the effective nonlinear conductivity a_e and the electric-field correlation function $\langle \mathbf{e}^4 \rangle$ of a metal–insulator mixture at the percolation threshold grow as $a_e \sim h^{-u_4}$, $\langle \mathbf{e}^4 \rangle \sim h^{-\epsilon_4}$ for $h \ll 1$, where $h = \sigma_2/\sigma_1$ is the ratio of linear conductivities. Dykhne's relations are used to derive important inequalities for the nonlinear conductivity and the critical exponents. The quantities a_e and $\langle \mathbf{e}^4 \rangle$ as functions of h were found by numerical modeling on a 100×100 grid and the critical exponents were calculated: $u_4 = \epsilon_4 = 1.33 \pm 0.05$. © *1995 American Institute of Physics.*

In the last few years an approach based on the ideas of fractal geometry[1] has made it possible to find new laws in the physics of disordered media. It has been found that the conducting channels in metal–insulator mixtures near the percolation threshold[2] are stochastic fractals.[1,3–5] This means that the current-carrying framework is geometrically a loose object with its own fractal dimension. Stochastic fractals differ from regular fractals by the existence of elements of randomness but on scales larger than the size b of the minimum nonuniformity and less than the correlation length ξ. They exhibit self-similarity properties and physical processes on them are described by power-law functions.[1] The effective linear conductivity of a metal–insulator mixture at the percolation threshold is a similarity function. Let σ_1 be the conductivity of the metal and σ_2 the conductivity of the insulator. For the critical concentration p_c corresponding to the threshold, a regime with mixed conductivity is realized:[5] The effective conductivity σ_e depends only on the ratio $h = \sigma_2/\sigma_1$. In the two-dimensional case ($p_c = 0.5$) σ_e as a function of h is given by the expression

$$\sigma_e = \sigma_1 h^{1/2}. \tag{1}$$

Equation (1) was first derived by Dykhne[6] from the dual symmetry of the current and field equations. He also showed that the electric field $\mathbf{e}$ in the mixed-conductivity regime can undergo enormous spatial fluctuations. The correlation function $\langle \mathbf{e}^2 \rangle$ of the field for $h \ll 1$ behaves as

$$\langle \mathbf{e}^2 \rangle / \mathbf{E}^2 \sim h^{-1/2}. \tag{2}$$

The correlation function of the squared field in the insulator phase behaves similarly.[6] The qualitative explanation of the behavior (2) is as follows. For small H a voltage drop occurs between the metal clusters on scales of the order of b, and the electric field in these sections is much stronger than the average field in the sample. In view of this circumstance, it should be expected that an effective intensification of nonlinear effects should occur near the percolation threshold. The electric conductivity of a nonlinear medium was studied in Refs. 5, 7, and 8. In these works, however, the nonlinear conductivity at the percolation threshold was not studied.

In the present paper we examine the two-dimensional, nonlinear, two-component medium at the percolation threshold. It is shown that in the case of cubic nonlinearity the effective nonlinear conductivity a_e grows at $h \ll 1$. Exact inequalities for a_e, the correlation functions of the fields, and the critical exponents are found. A direct numerical modeling is performed and a number of quantities are found as functions of the ratio of the conductivities of the linear medium.

Consider a disordered medium consisting of a metal and an insulator. The current density in the medium is described by the expression

$$\mathbf{j} = \sigma \mathbf{e} + a \mathbf{e}^2 \mathbf{e}, \tag{3}$$

where σ and a are random functions of the coordinates. For a two-component medium the conductivities can assume the values σ_1 and a_1 in the first medium and σ_2 and a_2 in the second medium. The current and field distributions are found from the equations

$$\operatorname{div} \mathbf{j} = 0, \quad \operatorname{curl} \mathbf{e} = 0. \tag{4}$$

We determine the effective linear conductivity σ_e and nonlinear conductivity a_e of the medium from the relations

$$\langle \mathbf{j}\mathbf{e} \rangle = \langle \sigma \mathbf{e}^2 \rangle + \langle a \mathbf{e}^4 \rangle = \sigma_e E^2 + a_e E^4, \tag{5}$$

where $\mathbf{E} = \langle \mathbf{e} \rangle$ is the volume-averaged electric field. To find the fields and currents in the nonlinear medium, we expand them in powers of the nonlinear parameter

$$\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 \ldots \tag{6}$$

To a first approximation in the nonlinearity parameter in expression (5) the cross term $\langle \sigma e_0 e_1 \rangle$ vanishes by virtue of Tellegen's theorem,[9] i.e., a_e is determined only by the first term in the expansion of the field.[8] Therefore, a_e depends on the distribution of the fields $\mathbf{e}_0$ in the linear medium:

$$a_e = \langle a \mathbf{e}_0^4 \rangle / E^4. \tag{7}$$

In what follows $\mathbf{e}$ is the electric field in the linear medium. It is assumed that the current density and the field are related to one another linearly by Ohm's law $\mathbf{j} = \sigma \mathbf{e}$ and that they satisfy Eqs. (4).

We shall show that to calculate the nonlinear characteristics of the medium, it is necessary to find the correlation function $\langle \mathbf{e}^4 \rangle$. The average of $\langle \mathbf{e}^4 \rangle_{1,2}$ for the components can be found from Dykhne's relations[6] in terms of the average for the entire system:

$$\langle \mathbf{e}^4 \rangle_1 = \frac{\sigma_2^2}{\langle \sigma^2 \rangle} \langle \mathbf{e}^4 \rangle, \quad \langle \mathbf{e}^4 \rangle_2 = \frac{\sigma_1^2}{\langle \sigma^2 \rangle} \langle \mathbf{e}^4 \rangle. \tag{8}$$

Using Eqs. (7) and (8), we obtain

$$a_e = \frac{\sigma_e^4}{\langle \sigma^2 \rangle} \left\langle \frac{a}{\sigma^2} \right\rangle \langle \mathbf{e}^4 \rangle / E^4. \tag{9}$$

Expression (1) can be obtained from (5) by making use of the property that the correlations for the dissipated energy $q=\mathbf{j}\cdot\mathbf{e}$ decouple: $\langle\mathbf{j}\cdot\mathbf{e}\rangle=\langle\mathbf{j}\rangle\cdot\langle\mathbf{e}\rangle=\sigma_e E^2$. In the case of the nonlinear conductivity, such a relation does not exist. However, we can derive a series of helpful inequalities, calculating the variance of the corresponding quantities.

Consider the variance of the local dissipation q. Using relation (9), we obtain the following relation after simple transformations

$$\langle(q-\langle q\rangle)^2\rangle=(a_e/\langle a/\sigma^2\rangle-\sigma_e^2)E^2\geqslant 0 \tag{10}$$

or

$$a_e/\sigma_e^2\geqslant\langle a/\sigma^2\rangle. \tag{11}$$

It follows from Eq. (11) that

$$a_e\geqslant(a_1h+a_2/h)/2. \tag{12}$$

The right-hand side of Eq. (12) has a minimum at $h=(a_2/a_1)^{1/2}$; i.e., the nonlinear conductivity always has a lower bound:

$$a_e\geqslant(a_1a_2)^{1/2}. \tag{13}$$

Calculating the variance of the quantity $\mathbf{e}^2$, we find

$$\langle\mathbf{e}^4\rangle/\mathbf{E}^4\geqslant\langle\sigma\rangle^2/\sigma_e^2. \tag{14}$$

We are interested in the behavior of the characteristics of the medium for $h\ll 1$. We introduce the critical exponents by the relations

$$\sigma_e\sim h^{u_2},\quad \langle\mathbf{e}^2\rangle\sim h^{-\epsilon_2},\quad a_e\sim h^{-u_4},\quad \langle\mathbf{e}^4\rangle\sim h^{-\epsilon_4}. \tag{15}$$

Dykhne found the values $u_2=1/2$ and $\epsilon_2=1/2$ [see Eqs. (1) and (2)]. Inequalities for the critical exponents follow from inequalities (11) and (14):

$$u_4\geqslant 1,\quad \epsilon_4\geqslant 1. \tag{16}$$

To determine the critical exponents, we performed direct numerical modeling of the dependence of the nonlinear conductivity and the correlation functions of the fields on the parameter h. The possibility of modeling the system at the percolation threshold for finite values of h is based on the following considerations. Every computer realization of the system on a finite lattice can be regarded as a fragment of a fractal. In the mixed conductivity regime the length[5] $l\sim h^{-\nu/2t}$, where $t/\nu=0.97$ (Refs. 1 and 2), plays the role of a characteristic scale of the nonuniformity. If the size of the fragment is greater than l, then effective averaging of the quantities being calculated will occur even inside the fragment. In our numerical experiments the maximum length of the edge of a square in the computational grid was equal to 100 lattice constants, so that the minimum admissible value of h was $\sim 10^{-4}$. The difference equations obtained from Eq. (4) were solved by an iteration method.[10] Computational difficulties arise because for small h the equilibration time of the system increases anomalously and, consequently, the number of required iterations increases. In view of this circumstance, the maximum size of the computational grid in our experiments did not exceed 100×100. To achieve a relative computational accuracy of $\sim 10^{-6}$ for the potential on such a grid, $\sim 10^5$ iterations were required. Averaging over ten realizations was performed in order to smooth out the fluctuations.

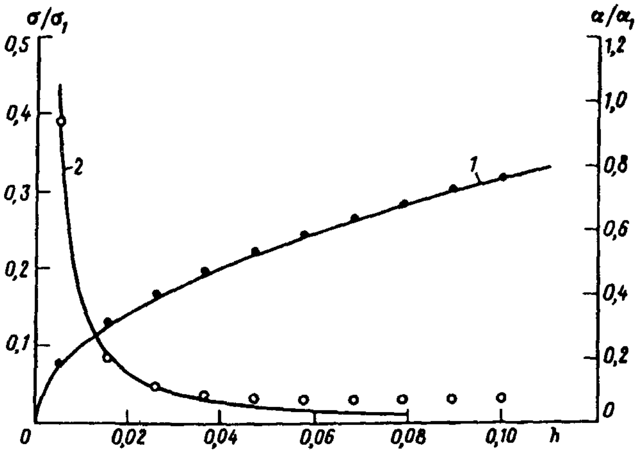

FIG. 1. σ_e and a_e as functions of h. ● — Numerical results for σ_e as a function of h on a 100× 100 grid; curve 1 — $\sigma_e=\sigma_1 h^{1/2}$; ○ — numerical results for a_e as a function of h; curve 2 — $a_e=a_1 h^{-1.33}$.

The known values of the critical exponents u_2 and ϵ_2 were used as a test. The quantities σ_e and a_e as functions of h are shown in Fig. 1. The calculations gave the following values for the critical exponents: $u_2=0.49\pm0.04$, $\epsilon_2=0.48\pm0.03$, $u_4=1.33\pm0.05$, and $\epsilon_4=1.33\pm0.05$.

In summary, we have shown that the nonlinear conductivity near the percolation threshold depends nontrivially on the ratio of the linear conductivities. This property indicates a new possibility for controlling the nonlinear conductivity of metal–insulator mixtures, which makes it possible to develop artificial nonlinear media with prescribed properties. We note that the characteristic field at which nonlinear effects are manifested also approaches zero as the percolation threshold is approached. In this study we have shown that the effective nonlinear conductivity depends on the ratio of the linear conductivities and we have calculated the critical exponents which determine this dependence.

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