Two-parameter regularization of crossover from directed to isotropic percolation

A. A. Luzhkov
St. Petersburg State Electrical Engineering University, 197376 St. Petersburg, Russia

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A continuous Hamiltonian describing, as special cases, ordinary and directed percolations is investigated on the basis of a renormalization scheme with two small parameters (the deviation from the corresponding critical dimension and the crossover index). A new physical realization of such a model is proposed — percolation phase transitions in systems without an inversion center, for example, percolation-type transitions into a ferroelectric phase in some disordered crystals and ceramics. © *1995 American Institute of Physics.*

Crossover between isotropic and directed percolations was actually analyzed in an earlier study,[1] but the effective Hamiltonian which incorporates both limiting cases was obtained just recently.[2] In the renormalization of this Hamiltonian in Ref. 2 a "generalized" scheme of minimal subtractions was employed. Despite the fact that this scheme has several shortcomings, which were noted by the authors themselves, it was possible to use it to investigate crossover of a pair correlation function of finite clusters in a one-loop approximation. However, the fact that in Ref. 2 the results were presented in the form of diagrams greatly complicates their physical interpretation.

This problem can be investigated, however, on the basis of a more easily understandable renormalization scheme with two small parameters ϵ and α, where ϵ is the deviation from the corresponding critical dimension, and α characterizes the crossover.[3–5] This method can be used, specifically, to investigate crossover of an infinite cluster (IC) and also to show that in the intermediate case it is necessary to introduce, in addition to the transverse R and longitudinal R_+ correlation lengths, the length R_- which characterizes the correlation in the negative direction (the reverse of the distinguished direction). In the isotropic case we have $R=R_+=R_-$ and for a strictly directed percolation[6] $R_+ \gg R$, $R_-=0$.

The Hamiltonian density of the mixed model[2] has the form

$$H=\varphi(\mathbf{x},t)\left[\tau-\Delta+a\frac{\partial}{\partial t}-b\frac{\partial^2}{\partial t^2}\right]\psi(\mathbf{x},t)+u(\varphi^2\psi-\psi^2\varphi), \tag{1}$$

where the dimension of the space transverse to the t axis is $d=4-\epsilon$, and the perturbation theory is supplemented by a rule for crossing out the "acausal" diagrams (for $b=0$ they drop out automatically). All parameters in Eq. (1) are assumed to be renormalized, $\tau=(p_c-p)/p_c$, where p_c is the percolation threshold. We shall consider a modified (in the sense of Ref. 4) version of Eq. (1), in which a is replaced a by $a\tau^\alpha$. In the case $0<\alpha<1/2$, by means of the transformations

$$\varphi(\mathbf{x},t)=\tau^{2A}\Phi(\mathbf{x}\tau^{A},t\tau^{B}),\quad B=2A-\alpha,\quad A=-\alpha/\epsilon, \tag{2}$$

and the same transformation of the field ψ via Ψ, we obtain a model which is equivalent to the standard model of directed percolation[6] but one for the fields Φ and Ψ, in which τ is replaced by $\tau_1=\tau^{1-2A}$. Although as a result of the transformation (2) the parameter b becomes $b_1=b\tau^{2A-2\alpha}$, for $\alpha<1/2$ b_1 will be irrelevant in the critical region for the same reasons that parameter b is irrelevant for the standard directed percolation, i.e., for model (1) with $a=$ const, $\tau\to 0$. We assume below that $\alpha\sim\epsilon$.

In discussing crossover, however, b must be included among the parameters on which the correlation functions depend. It is thus necessary to find the anomalous dimension of the parameter b in the basis of the fixed point of directed percolation.[6] We denote the renormalization constant for the parameter X by Z_X and its anomalous dimension by Γ_X. Obviously, Z_b is determined by the same diagrams as Z_τ within a replacement of the insertions τ in the interior lines by the insertions $b\omega^2$, where the "frequency" ω is the Fourier conjugate of t. Comparing the one-loop contributions, we obtain $Z_\tau Z=1+4K$ and $Z_bZ=1-K$, where $Z=1+K$ is the squared renormalization constant for both fields (K is the counter term). Hence it follows directly that $3\Gamma_b=-2\Gamma_\tau$.

The relation between the modified and standard models (1) makes it possible to find, just as in Ref. 4, the asymptotic representation of the Green's function $G(\alpha|\mathbf{x},t)$ with fixed but small τ and $x,t\to\infty$. We designate its Fourier transform as $G(\alpha|\mathbf{k},\omega)$. We have from Eq. 2

$$G(\alpha|\mathbf{k},\omega)=\tau^{-2A}G(k\tau^{-A},\omega\tau^{-B},\tau_1). \tag{3}$$

The Green's function on the right-hand side of Eq. (3) is the Green's function of the unmodified model (1):

$$G(k,\omega,\tau)=\tau^{-\gamma}g(kR,\ a\omega R_+,\ b\omega^2R^2\tau^{F}). \tag{4}$$

Here $R\sim\tau^{-\nu}$, $R_+\sim\tau^{-z\nu}$, where $\nu, z\nu\equiv\nu_+$ are the exponents for the correlation lengths in the transverse and longitudinal positive directions, respectively; $\gamma=\nu(2-\eta)$, η is Fisher's exponent, and $F=-\nu\Gamma_b=\epsilon/12$. The exponents ν and η are identical to the transverse exponents in Ref. 2. From Eqs. (3) and (4) we obtain the critical exponents of the modified model

$$\gamma(\alpha)=\gamma+2A(1-\gamma)=1+\frac{\epsilon+2\alpha}{6},\quad \nu(\alpha)=\frac{1}{2}+\frac{\epsilon+2\alpha}{16}, \tag{5}$$

$$\nu_+(\alpha)=B+\nu_+(1-2A)=1-\alpha+\frac{\epsilon+2\alpha}{12}.$$

For $x=0$, $t\to\infty$ the correlation properties of G are determined by the factor (see Ref. 1)

$$\Theta(t)\exp\left(-\frac{t}{2aR_+}\right)+\Theta(-t)\exp\left(\frac{2at}{b}\tau^{H}\right),\quad H=2\nu-\nu_+-F.$$

In the modified model with $t<0$ we have

$$G \sim \exp\left[-|t|\tau^{B}\frac{2a}{b_1}\tau_1^{H}\right] = \exp\left(\frac{t}{R_-}\right).$$

Obviously, R_- is the correlation length in the negative direction, and the corresponding exponent ν_- has the form

$$\nu_- = B + (1-2A)H - (2A-2\alpha) = \alpha + \frac{\epsilon + 2\alpha}{8}. \tag{6}$$

It is evident directly from Eqs. (5) and (6) that the modified theory indeed describes crossover from directed to isotropic percolation. Specifically, we have for the correlation lengths $R_+(\alpha) \ll R_+(0)$, $R_-(\alpha) \gg R_-(0)$. Comparing the exponents (5) and (6) with the $6-d$ expansion of the critical exponents of the isotropic percolation, we obtain $\gamma < \gamma(\alpha) < \gamma_0$, $\nu < \nu(\alpha) < \nu_0$, $\nu_0 < \nu_+(\alpha) < \nu_+$, where the zero subscript corresponds to the isotropic case. We note that the crossover index Δ introduced in Ref. 2 becomes $\Delta(\alpha) = (\frac{1}{2} - \alpha - (\epsilon + 2\alpha)/48)/\nu_+(\alpha)$ and for $\alpha = 0$ it is identical to the exponent obtained in Ref. 2 by analytic continuation to $b = 0$.

Below threshold the structure of an infinite cluster for directed percolation is determined by the formula (see Ref. 6) $G(x,t) = P^2\Theta(vt - x)$, $t \gg R_+$, where $P \sim \tau^{\beta}$, $v \sim R/R_+$. Taking into account (2) and (3), we obtain, in the modified model, $\beta(\alpha) = 1 - (\epsilon + 2\alpha)/6$. This exponent satisfies the standard equation $2\beta(\alpha) = d\nu(\alpha) + \nu_+(\alpha) - \gamma(\alpha)$. We also have $v \sim \tau^{X}$, where $X = \nu_+(\alpha) - \nu(\alpha) = \frac{1}{2} - \alpha + (\epsilon + 2\alpha)/48$.

For $\alpha = 0$ an infinite cluster is a narrow cone of possible directions of drift to infinity which deviate from the positive direction by allowed angles of the order of v. As α increases, crossover occurs to isotropic percolation, which consists of opening up of this cone, i.e., the allowed angle increases, since $v(\alpha) \gg v(0)$. When the limiting angle reaches 90°, a transition occurs abruptly to isotropic percolation (see Ref. 1).

This crossover is modeled theoretically[1] by the standard percolation clusters but with simple and directed bonds. Its natural (in our view) physical realization involves percolation phase transitions into a polar phase in disordered crystals and ceramics. In particular, in ferroelectrics such a transition should be accompanied by the formation of an infinite cluster from regions with the same direction of spontaneous polarization. If the conductivity of the polar phase is much greater than that of the paraphase, such a transition is accompanied by characteristic resistivity anomalies; this has been observed experimentally in, for example, Ref. 7. On the other hand, it is well known[8] that in media without a symmetry center, the conductivities in the forward (polar) and reverse directions can be different (for example, by several factors in measurements along and opposite to the direction of spontaneous polarization). Therefore, since in all elements of an infinite cluster the polarization vectors are directed in the same direction, the conductivities of the bonds in the forward and backward directions are different. In ceramics other mechanisms associated with the asymmetry of macrocontacts are also possible.[8]

In summary, we found that bonds which are parallel to the polarization vector are directed, and that for different ratios of the conductivities in the forward, backward, and transverse directions, a crossover of some sort is realized.

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