Quantum deformations of multi-instanton solutions in a twistor space

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We consider the quantum-group self-duality equation in the framework of the gauge theory on a deformed twistor space. Quantum deformations of the Atiyah–Drinfel'd–Hitchin–Manin and t'Hooft multi-instanton solutions are constructed. © *1995 American Institute of Physics.*

The quantum-group gauge theory has been considered in the framework of the algebra of local differential complexes[1–3] or as a noncommutative generalization of the fibre bundles over classical or quantum basis spaces.[4,5]

We prefer to use local constructions of the noncommutative connection forms or gauge fields as a deformed analog of the local gauge fields. In particular, the quantum-group self-duality equation (QGSDE) has been considered in a deformed 4-dimensional Euclidean space, and an explicit formula for the corresponding one-instanton solution has been constructed.[3] This solution can be treated as a q-deformation of the BPST-instanton.[6] Here we shall discuss quantum deformations of the general multi-instanton solutions.[7]

The conformal covariant description of the classical (ADHM) solution was considered in Ref. 8. We shall study the quantum deformation of this version of the twistor formalism. It is convenient to discuss first the deformations of the complex conformal group $GL(4,C)$, complex twistors, and complex linear gauge groups.

Let $R^{ab}_{cd}(a,b,c,d\ldots=1\ldots4)$ be the solution of the 4D Yang–Baxter equation which also satisfies the Hecke relation

$$RR'R=R'RR', \tag{1}$$

$$R^2=I+(q-q^{-1})R, \tag{2}$$

where q is a complex parameter. Note that the standard notation for these R-matrices is $R=\hat{R}_{12}, R'=\hat{R}_{23}$.[9]

Consider also the $SL_q(2,C)$ R-matrix

$$R^{\alpha\beta}_{\mu\nu}=q\,\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}+\varepsilon^{\alpha\beta}(q)\varepsilon_{\mu\nu}(q), \tag{3}$$

where $\varepsilon(q)$ is the deformed antisymmetric symbol.

Noncommutative twistors were considered in Ref. 10. We shall use the R-matrix approach to define a differential calculus on the deformed twistor space.

Let z^{α}_{a} and dz^{α}_{a} be the components of the q-twistor and their differentials

$$R^{\alpha\beta}_{\mu\nu}z^{\mu}_{a}z^{\nu}_{b}=z^{\alpha}_{c}z^{\beta}_{d}R^{dc}_{ba}, \tag{4}$$

$$z^{\alpha}_{a}dz^{\beta}_{b}=R^{\alpha\beta}_{\mu\nu}dz^{\mu}_{c}z^{\nu}_{d}R^{dc}_{ba}, \tag{5}$$

$$dz^{\alpha}_{a}dz^{\beta}_{b}=-R^{\alpha\beta}_{\mu\nu}dz^{\mu}_{c}dz^{\nu}_{d}R^{dc}_{ba}. \tag{6}$$

One can define also the algebra of partial derivatives ∂^{a}_{α}

$$R^{ab}_{cd}\partial^{c}_{\alpha}\partial^{d}_{\beta}=\partial^{a}_{\mu}\partial^{b}_{\nu}\,R^{\nu\mu}_{\beta\alpha}, \tag{7}$$

$$\partial^{a}_{\alpha}z^{\beta}_{b}=\delta^{a}_{b}\delta^{\beta}_{\alpha}+R^{\beta\mu}_{\alpha\nu}R^{da}_{cb}z^{\nu}_{d}\partial^{c}_{\mu}. \tag{8}$$

Consider the 4D deformed ε_q-symbol

$$R^{ba}_{fe}\varepsilon^{efcd}_{q}=-\frac{1}{q}\,\varepsilon^{abcd}_{q}. \tag{9}$$

The q-twistors satisfy the following identity:

$$\varepsilon^{abcd}_{q}z^{\beta}_{b}z^{\mu}_{c}z^{\nu}_{d}=0. \tag{10}$$

The $SL_q(2)$-invariant bilinear function of twistors has zero length in the projective 6D vector space

$$y_{ab}=\varepsilon_{\alpha\beta}(q)z^{\alpha}_{a}z^{\beta}_{b}=[P^{(-)}]^{dc}_{ba}y_{cd}, \tag{11}$$

$$(y,y)=\varepsilon^{abcd}_{q}y_{ab}y_{cd}=0. \tag{12}$$

Consider a duality transformation $*$ of the basic differential 2-forms[3]

$$*dz\;dz'=dz\;dz'P^{(+)}-dz\;dz'P^{(-)}, \tag{13}$$

where $P^{(\pm)}$ are the projection operators of $GL_q(4)$.[9] Note that the self-dual part $dz\;dz'P^{(+)}$ is proportional to

$$\varepsilon_{\alpha\beta}(q)dz^{\alpha}_{a}dz^{\beta}_{b}. \tag{14}$$

Let T^{i}_{k} be matrix elements of the $GL_q(N)$ quantum group

$$R_GTT'=TT'R_G, \tag{15}$$

where R_G is the R-matrix of $GL_q(N)$.

Quantum deformation of the $GL_q(N)$ gauge connection can be treated in terms of the noncommutative algebra for the components A^{i}_{k} of the connection 1-form[1,2]

$$(AR_GA+R_GAR_GAR_G)^{ikl}_{mnp}=0, \tag{16}$$

where $i,k,l,m,n,p=1\ldots N$. These relations generalize the anticommutativity conditions for components of the classical connection form.

The restriction on the quantum trace of the connection $\alpha=\mathrm{Tr}_qA=0$ is inconsistent with (16), but we can use the gauge-covariant relations $\alpha^2=0$, $\mathrm{Tr}_qA^2=0$ and $d\alpha=0$ (Ref. 3). The curvature 2-form $F=dA-A^2$ is q-traceless for this model.

Consider the explicit realization of this gauge algebra in terms of z,dz and the set B of additional noncommutative parameters

$$A^i_k(z,dz,B)=dz^\alpha_a A^{ai}_{\alpha k}(z,B). \tag{17}$$

The analogous realizations have been considered on the $GL_q(2)$ and $E_q(4)$ quantum spaces.[1-3] We shall treat the representation (17) as a local gauge field on the q-twistor space.

Let us consider the quantum deformation of the $GL(2)$ t'Hooft solution:[8]

$$A^\alpha_\beta=q^{-3}dz^\alpha_a(\partial^a_\mu\Phi)\Phi^{-1}\varepsilon^{\sigma\mu}(q)\varepsilon_{\sigma\beta}(q), \tag{18}$$

$$\Phi=\sum_i (X^i)^{-1}, \quad X^i=(y,b^i)=\varepsilon^{abcd}_q y_{ab}b^i_{cd}, \tag{19}$$

where b^i_{cd} are the noncommutative isotropic 6D vectors

$$db^i_{cd}=0, \quad (b^i,b^i)=0, \tag{20}$$

$$[y_{ab},X^i]=[b^i_{cd},X^i]=0. \tag{21}$$

The central elements X^i of the (B,z)-algebra do not commute with dz:

$$X^i dz^\alpha_a=q^2 dz^\alpha_a X^i. \tag{22}$$

We stress that A^α_β satisfies Eq. (16) and that its quantum trace is a $U(1)$-gauge field with zero field strength:

$$\mathrm{Tr}_q A=-q^{-3}d\Phi\Phi^{-1}, \quad \mathrm{Tr}_q dA=0. \tag{23}$$

The QGSDE for A^α_β is equivalent to the finite-difference Laplace equation for the function Φ on the q-twistor space

$$\Delta^{ba}\Phi(X^i)=\sum_i \Delta^{ba}\frac{1}{X^i}=0, \tag{24}$$

$$\Delta^{ba}\Phi=\frac{q}{1+q^2}\varepsilon^{\alpha\beta}(q)\partial^b_\beta\partial^a_\alpha\Phi=\left(\partial^{ba}+\frac{1}{2}y_{cd}\partial^{dc}\partial^{ba}\right)\Phi, \tag{25}$$

$$\partial^{ba}y_{cd}=[P^{(-)}]^{ab}_{dc}, \quad \partial^{ba}(X^i)^{-1}=-q^{-2}(X^i)^{-2}(b^i)^{ab}. \tag{26}$$

The ADHM-twistor functions of Ref. 7 can be connected with some $GL(N+2k)$ matrix function. Let us introduce the notation for indices of different types: $I,K,L,M=1\ldots N+2k$ and $A,B=1\ldots k$. The ansatz for the general self-dual $GL_q(N,C)$ field contains the deformed twistors $u(z)$ and $\tilde{u}(z)$

$$A^i_k=du^i_I\tilde{u}^I_k, \quad u^i_I\tilde{u}^I_k=\delta^i_k. \tag{27}$$

The commutation relations for the u and $\tilde{u}$ twistors are

$$(R_G)^{ik}_{lm}u^l_I u^m_K=u^i_L u^k_M \mathbf{R}^{LM}_{IK}, \tag{28}$$

$$\mathbf{R}^{KI}_{ML}\tilde{u}^L_i\tilde{u}^M_k=\tilde{u}^I_l\tilde{u}^K_m(R_G)^{ml}_{ki}, \tag{29}$$

$$\tilde{u}^I_l(R_G)^{li}_{mk}u^m_K=u^i_L\mathbf{R}^{IL}_{KM}\tilde{u}^M_k, \tag{30}$$

where the R-matrices for $GL_q(N,C)$ and $GL_q(N+2k,C)$ are used.

Consider also the linear twistor functions v and $\tilde{v}$

$$v_I^{A\alpha}=z_a^\alpha b_I^{aA},\tag{31}$$

$$\tilde{v}^{IA\alpha}=\tilde{b}^{IAa}z_a^\alpha.\tag{32}$$

We introduce the following condition for these functions:

$$v_I^{A\alpha}\tilde{v}^{IB\beta}=g^{AB}(z)\varepsilon^{\alpha\beta}(q),\tag{33}$$

where $g(z)$ is a nondegenerate $(k\times k)$ matrix with the central elements

$$g^{AB}(z)=\frac{q}{1+q^2}b_I^{aA}\tilde{b}^{IBb}y_{ab}.\tag{34}$$

The condition (33) is equivalent to the restriction on the elements of the B-algebra

$$[P^{(+)}]_{ab}^{cd}b_I^{aA}\tilde{b}^{IBb}=0.\tag{35}$$

We write the basic commutation relations of the B-algebra:

$$R_{cd}^{ab}b_I^{cA}b_K^{dB}=b_L^{aB}b_M^{bA}\mathbf{R}_{KI}^{ML},\tag{36}$$

$$\mathbf{R}_{LM}^{IK}\tilde{b}^{LAa}\tilde{b}^{MBb}=R_{cd}^{ab}\tilde{b}^{IBc}\tilde{b}^{KAd},\tag{37}$$

$$R_{cd}^{ab}b_I^{cA}\tilde{b}^{KBd}=\mathbf{R}_{IM}^{KL}\tilde{b}^{MBa}b_L^{aB}.\tag{38}$$

A formal permutation of the indices A and B is commutative. It is not difficult to give the relations between $b,\tilde{b}$ and z,dz.

Consider the new functions

$$\tilde{v}_{A\alpha}^I=g_{AB}(z)\varepsilon_{\alpha\beta}(q)\tilde{v}^{IB\beta},\tag{39}$$

where we use the inverse matrix of matrix (34).

Now one can construct the full quantum $GL_q(N+2k,C)$ matrices:

$$U=\begin{pmatrix}u_I^i\\ v_I^{A\alpha}\end{pmatrix},\quad \mathbf{U}^{-1}=\begin{pmatrix}\tilde{u}_i^I\\ \tilde{v}_{A\alpha}^I\end{pmatrix}.\tag{40}$$

The standard $GL_q(N+2k,C)$ commutation relations for these matrices contain Eqs. (28)–(30) and the relations for the functions v and $\tilde{v}$.

We write explicitly the orthogonality and completeness conditions for the deformed ADHM-twistors:

$$u_I^i\tilde{v}^{IA\alpha}=0,\tag{41}$$

$$v_I^{A\alpha}\tilde{u}_i^I=0,\tag{42}$$

$$\delta_K^I=\tilde{u}_i^Iu_K^i+\tilde{v}^{IA\alpha}g_{AB}(z)\varepsilon_{\alpha\beta}(q)v_K^{B\beta}.\tag{43}$$

The gauge-field algebra (16) for the deformed ADHM-ansatz (27) can be generated by a differential algebra on the $GL_q(N+2k,C)$ matrices $\mathbf{U}$, $\mathbf{U}^{-1}$, $d\,\mathbf{U}$ that contains the following relations:

$$\tilde{u}^l_i(R_G)^{ik}_{lm}du^l_K=du^k_L(\mathbf{R}^{-1})^{IL}_{KM}\tilde{u}^M_m\,, \tag{44}$$

$$du^i_L du^k_M(\mathbf{R}^{-1})^{LM}_{IK}=-(R^{-1}_G)^{ik}_{lm}du^l_I du^m_K\,. \tag{45}$$

These relations are consistent with the commutation relations (28)–(30).

The self-duality of the connection (27) follows from Eqs. (31), (32), (41)–(43),

$$dA^i_k-A^i_lA^l_k=du^i_I(\tilde{u}^I_lu^l_M-\delta^I_M)d\tilde{u}^M_k=-u^i_I\tilde{b}^{IAa}g_{AB}(z)\varepsilon_{\alpha\beta}(q)dz^\alpha_a dz^\beta_b b^{Bb}_M\tilde{u}^M_k\,. \tag{46}$$

This curvature contains the self-dual 2-form (14) only.

It should be stressed that all R-matrices of our deformation scheme satisfy the Hecke relation with the common parameter q. The other possible parameters of different R-matrices are independent. The case $q=1$ corresponds to the unitary deformations $(R^2=I)$ of the twistor space and the gauge groups. It is evident that the trivial deformation of the z-twistors is consistent with the nontrivial unitary deformation of the gauge sector and vice versa.

The Euclidean conformal q-twistors are a representation of the $U^*(4)\times SU_q(2)$ group. The anti-involution for these twistors has the following form:

$$(z^\alpha_a)^*=\varepsilon_{\alpha\beta}(q)z^\beta_bC^b_a\,, \tag{47}$$

where C is the charge conjugation matrix for $U^*(4)$. We can use the gauge group $U_q(N)$ in the framework of our approach.

An analogous construction can be considered for the real twistors and the gauge group $GL_q(N,R)$.

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