Long-time relaxation of current in a 2D weakly disordered conductor

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The long-time relaxation of the average conductance in a 2D mesoscopic sample is studied within the method recently suggested by Muzykantskiĭ and Khmelnitskiĭ and based on a saddle-point approximation to the supermatrix σ model. The resulting far asymptotic behavior is in perfect agreement with the result of renormalization group treatment by Altshuler, Kravtsov, and Lerner.

In a recent paper,[1] Muzykantskiĭ and Khmelnitskiĭ (MK) considered relaxation phenomena in disordered conductors in the framework of the supersymmetric σ-model approach. They suggested the nice idea that the long-time asymptotic behavior of the conductance $G(t)$ is governed by a nontrivial saddle point of the σ model. Their original goal was to reproduce, in a more direct way, the result of Altshuler, Kravtsov, and Lerner (AKL),[2] who found a logarithmically normal (LN) "tail" in the time dispersion in two and $(2+\epsilon)$ dimensions. However, MK found in 2D a different, power-law decay for moderately large times. They put forward a hypothesis that the LN asymptotic behavior could hold for longer times. Here I will show that this is indeed the case, and that this result can be obtained via the method developed by MK.

Following MK, I consider a 2D disk-shaped sample of a radius R. I assume unitary symmetry (broken time-reversal invariance) in course of the calculations. For systems of the orthogonal and symplectic symmetries, the treatment is completely analogous, and I simply present the corresponding results in the end of the paper. The problem can be described by the σ model with the action[3]

$$S=-\frac{\pi\nu}{4}\int d^2r\ \mathrm{Str}[D(\nabla Q)^2+2i\omega\Lambda Q]. \tag{1}$$

Here $Q(r)$ is 4×4 supermatrix field, D is the diffusion constant, ν the density of states, ω the frequency, Str denotes the supertrace, and $\Lambda=\mathrm{diag}(1,1,-1,-1)$. The saddle-point equation of MK reads:

$$\Delta_L\theta+\gamma^2\sinh\theta=0, \tag{2}$$

where $\theta(r)$ is the "noncompact angle" parametrizing the σ-model field $Q(r)$, Δ_L is the Laplacian operator, and $\gamma^2=i\omega/D$. It should be supplemented by the boundary conditions at the boundary with the leads

$$\theta|_{\text{leads}}=0 \tag{3}$$

and at the insulating boundary

$$\nabla_n \theta|_{\text{insulator}}=0, \tag{4}$$

where ∇_n denotes the normal derivative.

We can consider the two leads attached to the disk boundary to be of almost semicircular shape, with relatively narrow insulating gaps between them. Then we can approximate the boundary conditions by using Eq. (3) for the entire boundary, as was done by MK. In fact, in view of the logarithmic dependence on R of the saddle-point action (see below), the result should not depend, to the leading approximation, on the specific shape of the sample and the leads attached. With the rotationally invariant form of the boundary condition, the minimal action corresponds to the situation that the function θ depends on the radius r only. We thus get the radial equation

$$\theta''+\theta'/r+\gamma^2 \sinh \theta=0; \quad 0\leqslant r\leqslant R \tag{5}$$

(the prime denotes the derivative d/dr), with the boundary conditions:

$$\theta(R)=0, \tag{6}$$

$$\theta'(0)=0. \tag{7}$$

Condition (7) follows from the requirement of analyticity of the field at the center of the disk.

Assuming that the characteristic values of θ satisfy the condition $\theta\gg 1$ (we will find below the corresponding restriction on the time t), one can replace $\sinh \theta$ by $e^{\theta}/2$. Equation (5) can be then easily integrated, and its general solution reads:

$$e^{\theta(r)}=\frac{4C_1^2}{\gamma^2}\frac{C_2 r^{C_1-2}}{(C_2 r^{C_1}+1)^2}, \tag{8}$$

with two integration constants C_1 and C_2. To satisfy the boundary condition (7), we have to choose $C_1=2$. Furthermore, the above assumption $\theta(0)\gg 1$ implies that $4C_2/\gamma_2\gg 1$. Therefore, the second boundary condition (6) is satisfied if $C_2\simeq(4/\gamma R^2)^2$, and the solution can be written in the form

$$e^{\theta(r)}\simeq[(r/R)^2+(\gamma R/4)^2]^{-2}. \tag{9}$$

Using now the self-consistency equation of MK,

$$2\pi\int_0^R dr\ r(\cosh\ \theta-1)=t/\pi\nu, \tag{10}$$

one finds $\gamma^2=8\pi^2\nu/t$. Finally, the action

$$S\simeq\pi^2\nu D\int dr\ r(\theta'^2-\gamma^2 e^{\theta}) \tag{11}$$

is equal on the saddle point (9) to

$$S\simeq 8\pi^2\nu D\ \ln(t\Delta), \tag{12}$$

where $\Delta = 1/\nu\pi R^2$ is the mean level spacing. Equation (11) is exactly the same as the result of MK. The above treatment is valid provided that $\theta'(r) < l^{-1}$ on the saddle point solution, which is the condition of applicability of the diffusion approximation (here l is the mean free path). In combination with the assumption $\theta(0) \gg 1$, this means that $1 \ll t\Delta \ll (R/l)^2$.

Now I consider the ultralong-time region, $t \gg \Delta^{-1}(R/l)^2$. In order to support the applicability of the diffusion approximation, we should search for a function $\theta(r)$ minimizing the action with an additional restriction $\theta' \leq A l^{-1}$. Here A is a parameter of the order of unity, which can not be determined within the diffusion approximation. We will see, however, that the saddle-point action depends on l through $\ln(R/l)$ only, and thus does not depend on A in the leading order, so that we can set $A=1$. Since the derivative has a tendency to increase in the vicinity of $r=0$, the restriction can be implemented by replacing the boundary conditions (7) by $\theta'(r_*)=0$, where the parameter r_* will be specified below. The solution now reads

$$e^{\theta(r)} = \frac{(r/R)^{C-2}}{[(r/R)^C + (C+2/C-2)(r_*/R)^C]^2}; \quad r_* \leq r \leq R. \tag{13}$$

The function $\theta(r)$ is meant as being constant within a neighborhood $|r| \leq r_*$ of the disk center. The condition $\theta' \leq l^{-1}$ yields $r_* \sim lC$. It is important to note that the result does not depend on details of the cut-off procedure. For example one gets the same results if one chooses the boundary condition in the form $\theta'(r_*) = 1/l$. The crucial point is that the maximum derivative θ' should not exceed $1/l$. The constant C is to be found from the self-consistency equation (10), which can be reduced to the following form:

$$\left(\frac{R}{r_*}\right)^C = \frac{2t}{\pi^2 \nu R^2} \frac{C^2}{C-2}. \tag{14}$$

Neglecting corrections of the form $\ln(\ln \cdot)$, we find

$$C \simeq \frac{\ln(t\Delta)}{\ln(R/r_*)} \simeq \frac{\ln(t\Delta)}{\ln(R/l)}. \tag{15}$$

The action (11) is then equal to

$$S \simeq \pi^2 \nu D (C+2)^2 \ln(R/r_*) \simeq \pi^2 \nu D \frac{\ln^2[t\Delta(R/l)^2]}{\ln(R/l)}. \tag{16}$$

For the orthogonal and symplectic ensembles, the saddle-point equation (5) has the same form, with the only difference that the action (11) is multiplied by the factor $\beta/2$, where $\beta = 1, 2, 4$ for the orthogonal, unitary and symplectic symmetries respectively. Combining Eqs. (11) and (16), we thus get for the long-time asymptotic behavior of the average conductance $G(t) \sim e^{-S}$ in all three symmetry cases:

$$G(t) \sim (t\Delta)^{-2\pi\beta g}, \quad 1 \ll t\Delta \ll (R/l)^2; \tag{17}$$

$$G(t) \sim \exp\left\{ -\frac{\pi\beta g}{4} \frac{\ln^2(t/g\tau)}{\ln(R/l)} \right\}, \quad t\Delta \gg (R/l)^2, \tag{18}$$

where $g=2\pi\nu D$ is the dimensionless conductance per square in 2D and τ is the mean free time.

The far asymptotic behavior [Eq. (18)] is of the LN form and very similar to that found by AKL (see Eq. (7.8) in Ref. 2). It differs only by the factor $1/g$ in the argument of $\ln^2$. It is easy to see however that this difference disappears if one does the last step of the AKL calculation to a better accuracy. Let us consider for this purpose the intermediate expression of AKL [Ref. 2, Eq. (7.11)]:

$$G(t)\propto-\frac{\sigma}{\tau}\int_0^\infty e^{-t/t_\phi}\exp\left[-\frac{1}{4u}\ln^2\frac{t_\phi}{\tau}\right]\frac{dt_\phi}{t_\phi}, \tag{19}$$

where $u\simeq(1/2\pi^2\nu D)\ln(R/l)$ in the weak localization region in 2D, which we are considering. Evaluating the integral (19) by the saddle point method, we find

$$G(t)\sim\exp\left\{-\frac{1}{4u}\ln^2\frac{2ut}{\tau}\right\},\quad G(t)\sim\exp\left\{-\frac{\pi g}{4}\frac{\ln^2(t/g\tau)}{\ln(R/l)}\right\}, \tag{20}$$

where we have kept only the leading term in the exponent. Equation (20) is in exact agreement with Eq. (18) for $\beta=1$ (AKL assumed orthogonal symmetry of the ensemble). Therefore, the supersymmetric treatment confirms the AKL result and also establishes the region of its validity. It is instructive to represent the results in terms of the superposition of simple relaxation processes with mesoscopically distributed relaxation times t_ϕ:

$$G(t)\sim\int\frac{dt_\phi}{t_\phi}e^{-t/t_\phi}P(t_\phi). \tag{21}$$

Then from Eqs. (17), (18) we have the distribution function $P(t_\phi)$

$$P(t_\phi)\sim\begin{cases}(t_\phi/t_D)^{-2\pi\beta g}, & t_D\ll t_\phi\ll t_D\left(\frac{R}{l}\right)^2\\ \exp\left\{-\frac{\pi\beta g}{4}\frac{\ln^2(t_\phi/\tau)}{\ln(R/l)}\right\}, & t_\phi\gg t_D\left(\frac{R}{l}\right)^2,\end{cases} \tag{22}$$

where $t_D\simeq R^2/D$ is the time of diffusion through the sample.

For completeness, we list also the results for quasi-1D and 3D systems. For a quasi-1D sample (wire) of length L (which is assumed to be much shorter than the localization length $\xi=2\beta\pi\nu D$) the asymptotic behavior is given by

$$G(t)\sim\exp\left\{-\frac{\beta\pi\nu D}{L}\ln^2(t\Delta)\right\},\quad t\Delta\gg1 \tag{23}$$

[for $\beta=2$ this is just Eq. (16) of MK]. Equation (23) is valid up to the exponentially large time $t\sim\exp(L/l)$, (Ref. 1). It is interesting to note that Eq. (23) has essentially the same form as the asymptotic formula for $G(t)$ found by Altshuler and Prigodin[4] for the *strictly* 1D sample with a length much *exceeding* the localization length:

$$G(t)\sim\exp\left\{-\frac{1}{L}\ln^2(t/\tau)\right\}. \tag{24}$$

If we replace in Eq. (24) the 1D localization length $\xi=2l$ by the quasi-1D localization length $\xi=2\beta\pi\nu D$, we reproduce the asymptotic expression (23) (up to a normalization of t in the argument of $\ln^2$, which does not affect the leading term in the exponent for $t\to\infty$). This leads us to make the following two conclusions. First, this confirms once more the general conjecture[5] that the statistical properties of smooth envelopes of the wave functions in 1D and quasi-1D samples are identical. Second, this shows that the asymptotic "tail" (23) in the metallic sample is indeed due to "quasi-localized" eigenstates, as has been conjectured.[1,4,6,7]

In 3D, the analysis proceeds along the same line as for the ultralong-time region in 2D. This is very similar to what has been done by MK in their consideration of the 3D case, so that I am presenting a brief sketch of the derivation only. I consider the saddle-point equation in the spherically symmetric form

$$\theta''+2\theta'/r+\gamma^2\sinh\theta=0,\quad \theta(R)=0,\quad \theta'(r_*)=0. \tag{25}$$

At $r\gg r_*$ the last term on the left-hand side of Eq. (25) can be neglected,[1] and $\theta(r)\simeq C(R/r-1)$. The maximum derivative θ' is reached at $r\sim r_*$, so that the condition $\theta' l\leqslant A$ with $A\sim 1$ implies $r_*\sim(CRl)^{1/2}$. The self-consistency equation

$$4\pi\int r^2dr(\cosh\theta-1)=t/\pi\nu$$

then yields $r_*^3 e^{CR/r_*}\sim t/\nu$, and consequently,

$$C\sim\frac{l}{R}\ln^2\left[\frac{t}{\tau(k_Fl)^2}\right];\quad r_*\sim l\ln\left[\frac{t}{\tau(k_Fl)^2}\right].$$

Multiplying Eq. (25) by θ' and integrating it from r_* to R, we get $\gamma^2\exp\theta(r_*)\sim 1/l^2$. Finally, the saddle-point action is estimated as

$$S(t)\simeq 2\pi^2\nu D\beta\int r^2dr(\theta'^2-\gamma^2e^\theta)\sim\nu D\frac{r_*^3}{l^2}\sim\nu Dl\ln^3\left[\frac{t}{\tau(k_Fl)^2}\right]. \tag{26}$$

Since the action (26) is proportional to l, the numerical prefactor in (26) cannot be found within the diffusion approximation. We get therefore the following contribution to the average conductance $G(t)$,

$$G(t)\sim\exp\{-S(t)\},\quad S(t)\sim(k_fl)^2\ln^3\left[\frac{t}{\tau(k_fl)^2}\right], \tag{27}$$

which dominates over the usual cooperon-induced term,[1,2] $G(t)\sim\exp(-t/t_D)$, at $t\gg(k_fl)^2t_D\ln^3(t_D/\tau)$.

Let us now discuss the approximations involved and the limitations of the method. We have employed the saddle-point method for the σ model and found that for long enough time in 2D (and also in 3D the solution in the vicinity of its "center" $r=0$ violates the condition of applicability of the diffusion approximation, $\theta' l\ll 1$. We assumed then that the correct result can be found by allowing only those configurations of the Q field which satisfy the restriction $\theta' l\leqslant A$ with $A\sim 1$. This is a natural extension of cutting off the diffusion and cooperon momenta at $q\sim l^{-1}$, the procedure commonly used in perturbation theory. To implement the above restriction, we shifted the boundary condition from the singular point $r=0$ to $r=r_*$. Though the numerical constant $A\sim 1$ cannot be specified in this way, it does not affect the long-time behavior of $G(t)$ in 2D case. At the same time, it enters the prefactor (in front of the cube of the logarithm in the exponent in the 3D case. It would be desirable, of course, to justify rigorously the above cutoff procedure, and to determine the numerical coefficient in the action in 3D. For this purpose, one should go beyond the long-wavelength σ-model approximation and develop a generalization of the method valid also on a scale $r \lesssim l$. A step forward in this direction was done in a very recent paper.[8]

We note in conclusion that the long-time asymptotic behavior found for the average conductance $G(t)$ has a form very similar to the asymptotic behavior of the distribution function $P(\rho)$ of the local density of states (LDOS).[9] In both cases, the result is of the LN form in quasi-1D and 2D, and of a somewhat different (though very similar) "log-cube-exponential" form in 3D. As in the case of the LDOS distribution,[9] we have found a perfect agreement with the result of a renormalization group (RG) treatment[2] in 2D. I believe this agreement between the RG and supersymmetric treatments of $G(t)$ and $P(\rho)$ to be of considerable conceptual importance. It is of a nontrivial nature, since the supersymmetric solution heavily relies on the noncompact structure of the supersymmetric σ-model manifold and is dominated by the large values $\theta \gg 1$ of the "noncompact angle" θ, whereas the RG treatment is just a resummation of the perturbation expansion and does not distinguish between the compact and noncompact versions of the σ model. This agreement provides strong support to other results obtained within the RG approach in the weak localization region and in the vicinity of the Anderson transition.[2] On the other hand, we see that the supersymmetry method is in many cases able to reproduce the results of the RG treatment in a more elegant way. Furthermore, it is not restricted like the RG to the spatial dimension $d=2$ and can be successfully applied to quasi-1D and 3D systems as well. Besides the study of conductivity relaxation $G(t)$ and the LDOS distribution $P(\rho)$ discussed above, I would like to mention in this context the recent progress in understanding of the statistical properties of eigenfunctions.[5,10,11] Seeing that the two approaches are in amazingly good agreement, we can (depending on the problem considered) use any of them or even combine them to complete our understanding of the properties of mesoscopic disordered systems.

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