"Wild sphere" of correlations

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The fine structure of the spatial dispersion of a two-point correlation function of the fluctuating field at the critical point was investigated. It was shown numerically that the formation of a "filamentary" large-scale spatial structure of this field results in the fact that the Fourier transform of the correlation function of the field is strongly irregular (fractal) at low momenta. It was determined that such a fine structure of the correlation function is the physical reason for the nonintegral (anomalous) dimension of the smooth correlation function which is described in the analytic theory by Fisher's critical exponent η.

INTRODUCTION

The correlation functions of random fields are one of the most important characteristics of fluctuating systems. They are defined as ensemble averages of the products of the field variables at different points in space $G(\mathbf{r}-\mathbf{r}')=\langle\varphi(\mathbf{r})\varphi(\mathbf{r}')\rangle$. Proceeding from the hypothesis that such quantities exhibit the self-averaging property, it is ordinarily assumed that these averages are approximated well by the solutions of truncated systems of linked equations which relate functions of different orders.[1-4] The solutions obtained in this manner must be verified, i.e., the spatial correlations must indeed decrease so rapidly (exponentially) that they ensure the desired self-averaging. One of the physically most interesting cases — the case of a system at the point of a second-order phase transition — is not, however, such a system. Here the correlation functions decay in a power-law fashion, and the corresponding systems of equations cannot be truncated.[3]

Nonetheless, the approach based on the rough renormalization-group (RG) description gives very good approximations for the critical exponents, specifically, for Fisher's exponent η, which describes the asymptotic behavior of the Fourier transform $G(q) \propto q^{\eta-2}$ of the two-point correlation function $G(\mathbf{r}-\mathbf{r}')$ at the critical point. In the renormalization group the appearance of a nonintegral "anomalous dimension" $0\leq\eta\leq 1/4$ is attributed to the generation of nonlocal renormalizations of the Ginzburg–Landau functional.[1-5] Naturally, it does not reduce only to a change in the asymptotic behavior, but it also affects the entire structure of $G(\mathbf{q})$ at the critical point.[5]

The change in the correlation functions in the normalization group occurs in an abstract space of the parameters of the function space and the fictitious "renormalization group time." Despite the fact that the fluctuation theory of phase transitions has already existed for a long time,[1] it is important that the physical mechanisms responsible for the generation of anomalous dimensions be explained. This letter addresses this problem.

MODEL AND RESULTS

We base our analysis on the solution of the kinetic equation for the fluctuating field of the order parameter. This procedure is, to a certain extent, an alternative to reduction of the description in the renormalization-group method. When this approach is executed systematically, it is assumed that the Ginzberg–Landau functional derived from the microscopic analysis (see, for example, Refs. 4 and 5)

$$\mathscr{H}[\varphi]=\int d^d r\left[\frac{1}{2}(\nabla\varphi)^2+F(\varphi)\right] \tag{1}$$

(in a d-dimensional space) is unchanged, while all effects of the renormalizations of its fluctuations are obtained "automatically" by solving the kinetic equation for the order parameter

$$\partial\varphi(\mathbf{r},t)/\partial t=c\Delta\varphi-\partial F/\partial\varphi+\xi(\mathbf{r},t), \tag{2}$$

provided that the condition $\varphi(\mathbf{r})=0$ is satisifed at the instant $t=0$ of the introduction of the random source $\xi(\mathbf{r},t)$:

$$\langle\xi(\mathbf{r},t)\rangle=0;\quad \langle\xi(\mathbf{r},t)\xi(\mathbf{r}',t)\rangle=D\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{3}$$

Equation (2) and the renormalization-group equation are solved approximately, but at the phenomenological level the results of the two approaches must be identical.[6]

An exact calculation, even one which uses the local free energy $F(\varphi)$, in the critical region requires an infinite numerical array and cannot be performed. However, the important points for understanding the characteristic features of the critical behavior can be determined by using the following approximation.

We assume that an infinite number of small-scale fluctuating modes has already been summed and taken into account by the renormalization of the effective local form $F(\varphi)\to F_{\text{eff}}(\varphi)$. In this context, $F_{\text{eff}}(\varphi)$ can be calculated adequately for the case at hand by using the local version of the exact renormalization-group equation:[7]

$$\partial f/\partial l=\hat{R}f=df+\Delta_{\varphi\varphi}f-\frac{d-2}{2}\nabla_{\varphi}\varphi-(\nabla_{\varphi}f)^2, \tag{4}$$

assuming that $F_{\text{eff}}(\varphi)$ coincides with its stationary point, $f^*=F_{\text{eff}}(\varphi)$. At the same time, it should not be forgotten that the terminology $F_{\text{eff}}(\varphi)$ is in itself very conditional, since it reduces the description of a system in a $(d\cdot n)$-dimensional space $\{\varphi,\ \nabla\varphi\}$ (where n is the number of components of the order parameter) to a purely local (n-dimensional) space $\{\varphi\}$. In what follows we shall take into account the subspace $\{\nabla\varphi\}$.[6] For simplicity, we restrict the analysis to the even $f(\varphi)=f(\varphi^2)$ and scalar ($n=1$) version of the model.

In a wide temperature range (even outside strict scale invariance) the function $f(l)=\Sigma g_{2k}(l)\varphi^{2k}$ has an infinite number of nonzero coefficients g_{2k}. Subtracting the renormalization of the temperature T_c (i.e., $g_2\varphi^2$), the corresponding effective free energy $F_{\text{eff}}(\varphi^2)$ is monotonic and anomalously flat compared to the φ^4 model. In view of its monotonicity, the nonlinear excitations of the nuclear type with a first-order phase transition (PT1) are disadvantageous. Nonetheless, here such excitations are long-lived formations, approximating the real distributions $\varphi(\mathbf{r})$ which are close to satisfying the

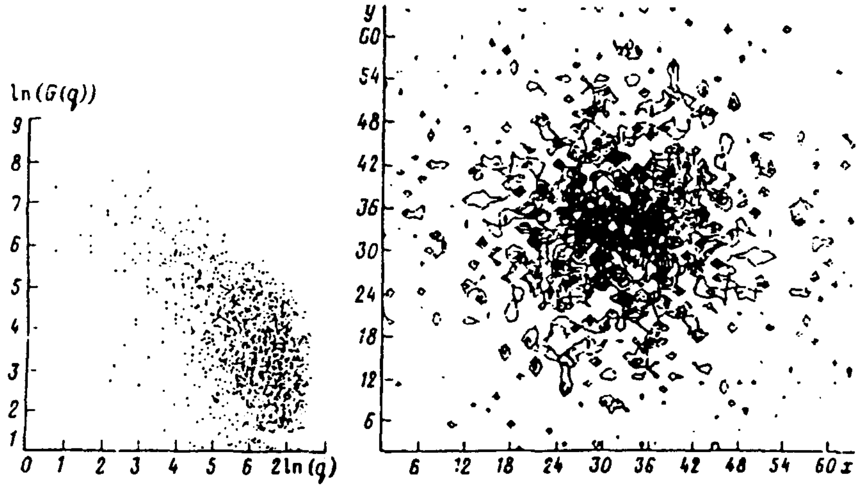

FIG. 1. Map of contour lines of the Fourier transform of the correlation function $G(q)$. The figure on the left-hand side displays in double-logarithmic coordinates the distribution of the values of G as a function of the absolute value of the radius $q=(q_x^2+q_y^2)^{1/2}$.

condition $\delta\mathcal{H}/\delta\varphi=0$ and therefore relax extremely slowly. Just as for a PT1, these formations appear spontaneously from an arbitrary fluctuating field and in the process form a filamentary structure.[8-10]

It can be shown that when the stationary functional f^* is substituted as an effective free-energy functional at the critical point into the equation for $\partial\varphi/\partial t$, the relaxation of the field $\varphi(\mathbf{r})$ leads to a simple scale transformation of the distribution $\exp\{-\mathcal{H}[\varphi]\}$ in time.[7] The system produces increasingly larger organized structures. The correlation functions not only acquire a scaling form (i.e., on the average, they are power-law functions of the distance), but also are self-averaging quantities because of this acquisition; i.e., they retain a memory about the structuring of the field.

For practical calculations the function $F_{\text{eff}}(\varphi^2)$ can be used both in the form of a numerical array found from Eq. (4) and in the form of a function approximated by Padé approximants:

$$F_{\text{eff}}(\varphi^2)\approx(a_1+a_3\varphi^2+a_5\varphi^4+\ldots)/(a_2+a_4\varphi^2+a_6\varphi^4+\ldots). \tag{5}$$

Even the lowest nontrivial approximation $a_1=a_3=a_{2k>4}=0$, a_5, a_2, $a_4\neq 0$ gives a relative order of accuracy of $10^{-3}-10^{-2}$. We have performed the corresponding numerical calculations. They yielded the structure of the Fourier transform of the correlation function represented by the contour lines in Fig. 1 [for one typical realization of the distribution $\varphi(\mathbf{r})$]. As expected, because of the anomalously slow decay of the correlations, $G(q)$ preserves a memory of the structuring of the field. The irregularity of each contour line [constant $G(q)=\bar{G}$] is comparable to the average value of the radius q

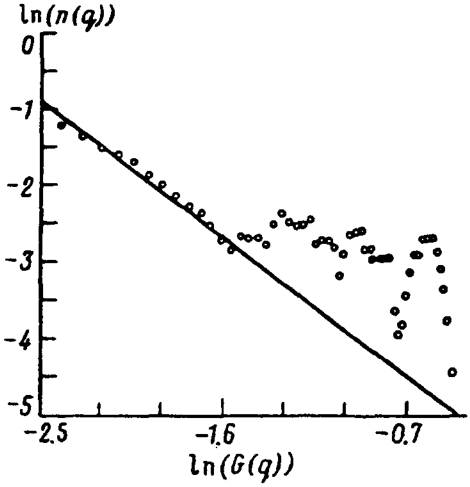

FIG. 2. Relation between the density of states $n(q)$ and the correlation function $G(q)$. The straight line corresponds to the function $\ln[G(q)]=\ln[n(q)]+\text{const}$.

which corresponds to the given $\bar{G}$. This is seen clearly on the left-hand side of Fig. 1, on which this sample of values of $G(q)$ is given as a function of the modulus $q=(q_x^2+q_y^2)^{1/2}$.

The effect of the fractal nature on the structure of $G(q)$ can be characterized as follows. We note that in the mean-field theory $G(q) \propto 1/q^2$. Since the length of the circumference with this value of $G(q)=\bar{G}$ is proportional to q, the number of states $n(q)$ with given $G(q)=\bar{G}$ should be proportional to $G(q)^{-1/2}$. The real dependence of $n(q)$ on $G(q)$ is reproduced on a double logarithmic scale in Fig. 2. For large values of q this function is a straight line given by $\ln(G)=-2\ln(n)+\text{const}$ and for small values of q it is strongly irregular. The concrete picture of this irregularity is specific to each concrete realization. However, the function $n[G(q)]$ has a sign-definite deviation from the desired straight line. This is characteristic and important. It means that the dimension of the lines $G(q)=\bar{G}$ for small q is always high compared to the "naive dimension," which is equal to 1.

The function $G(q)$, filtered with respect to the low momentum scales, must be the same as in the renormalization-group theory, i.e., $G(q) \propto 1/q^{2-\eta}$.[1-5] Figure 3 and the inset in this figure show the filtered G function in the same notation as the initial function in Fig. 1. Moreover, the straight lines which in double-logarithmic coordinates correspond to the mean-field function $G(q) \propto 1/q^2$ (dashed line) and the scaling formula are shown on the left-hand side of the figure. The quantity η was chosen to be equal to 1/4, i.e., the known value for the dimension[1-5,11] $d=2$.

The structuring of $G(q)$ ultimately reflects the effect of the gradient terms in the energy, since the (quasi-) low-dimensionality of the excitations lowers the contribution from some of the components of the gradients, which are small along the folds of the

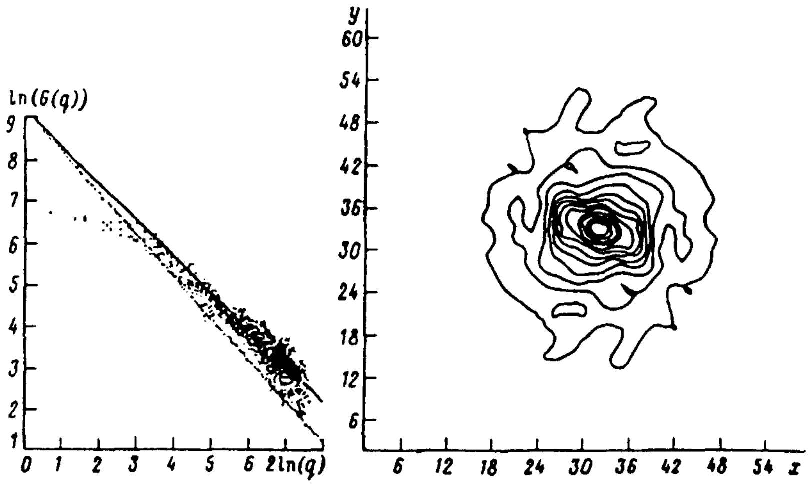

FIG. 3. Same as Fig. 1 for the function $G(q)$ smoothed at low momenta. The dashed straight lines and the solid line on the right-hand side of the figure represent the functions $\ln(G)=-2\ln(q)+\text{const}$ and $\ln(G)=-7\ln(q)/4+\text{const}$, respectively.

density[10] $\varphi(\mathbf{r})$. This effect decreases the effective dimension of the fluctuations and ultimately leads to an anomalous dimension of G.

The numerical experiments show that a similar structuring of the field occurs in the three-dimensional space. The lines of constant G transform here into densely irregular (down to $q=0$) surfaces which are similar to the so-called "wild sphere" described by mathematicians.[12]

In this context, it should be noted that similar fractal properties of phase transitions were also discovered numerically in the past for Ising clusters on the basis of approaches which are alternatives to the standard theory based on the order-parameter field.[13,14] The results obtained by the Monte Carlo method and on the basis of the Ginzburg–Landau functional confirm and supplement one another. We were also able to trace how the irregularity of the fluctuating field transforms into rough characteristics of the analytical theory, for example, into the anomalous dimension η. Specifically, it can be expected that since a lowering of the energy with $d=3$ is associated with (quasi-) two-dimensional fragments of the structure, the corresponding change in the dimension manifests itself in a weaker manner. This property should be closely associated with the lower value of η, $\eta\approx0.03$, for $d=3$ (Refs. 1–5). We shall check this assertion in a future work.

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