Radiative mechanism of lepton bound-state production

A. I. Akhiezer and N. P. Merenkov[1)]

"Kharkov Institute of Physics and Technology" National Science Center, 310108 Kharkov, Ukraine

(Submitted 2 November 1995)

Pis'ma Zh. Éksp. Teor. Fiz. **62**, No. 11, 839–843 (10 December 1995)

The radiative mechanism of lepton bound-state formation is investigated. It results in real photon emission and bound-state production in a free lepton interaction. The cross sections for the corresponding processes are derived, and some figures given for the case of positronium formation in free electron–positron collisions. © *1995 American Institute of Physics.*

Various bound states of oppositely charged leptons may be produced as a result of the electromagnetic interaction. The same interaction leads to the formation of bound states of leptons with a proton or antiproton. Since the theory of all these states is based on the Coulomb interaction, it may be constructed by the same method.

We shall demonstrate the theory of the formation of such states first of all for the example of positronium. Note that the quantum states and decay mechanisms of positronium are well known in theory.[1,2] Moreover, the mechanism of the radiationless capture of the positron by an atomic electron is well investigated (see Refs. 3–6 and references therein). Meanwhile there is another mechanism of positronium formation, namely the mechanism of radiative capture. It describes the formation of a positrionium atom in an electron–positron collision with the resulting emission of a real photon. In this case both the initial particles (electron and positron) are free. This mechanism is in essence analogous to the radiative capture of a proton by a neutron (first investigated by Fermi[7]), which results in real photon emission and deuteron production. The study of this mechanism is of great interest, especially in connection with the investigation of electron–positron plasma properties.[8]

The theoretical description of the radiative capture is straightforward, as it is based on the general methods of quantum-electrodynamics perturbation theory. We deal here with the radiative transition of particles from the free state to the bound one. Therefore, this mechanism may be regarded as a peculiar inverse photoeffect. It may be used for investigating the formation of the different bound states. In this paper we consider the mechanism of the radiative capture of leptons with arbitrary masses which results in real photon radiation and bound-state formation.

In Section 1 we give the formulas for corresponding matrix elements and cross sections, and in Section 2 the case of positronium formation is investigated.

1. In this section we consider the transition of a system containing two oppositely charged free particles into the bound state. This transition must be accompanied by the emission of a real photon. For the sake of definiteness we shall speak about the transition of an electron and a muon to muonium (but the all derived formulas can be applied to any radiative transition process of two oppositely charged particles interacting by the Coulomb law):

$$\mu^+(p_+)+e^-(p_-)\to\gamma(k)+(\mu^+e^-)(p), \tag{1}$$

where corresponding 4-momenta are indicated in the brackets.

The differential probability of the radiative transition (1) per unit volume per unit time may be written as

$$dw=\frac{dW}{TV}=\frac{e^2\omega d\Omega|\bar{M}|^2}{2(2\pi)^2}, \tag{2}$$

where ω is the energy and $d\Omega$ is the solid angle element of the photon. The reduced matrix element in Eq. (2) will be defined as

$$\bar{M}=\int d^3r\Psi_f^*(\mathbf{r})\left(\frac{\mathbf{a}\nabla}{\mathrm{m}}e^{i\mathbf{k}_1\mathbf{r}}+\frac{\mathbf{a}\nabla}{\mu}e^{-i\mathbf{k}_2\mathbf{r}}\right)\Psi_i(\mathbf{r}), \tag{3}$$

where $\mathbf{r}$ is the radius vector of the relative motion, $\mu(m)$ is the muon (electron) mass, $\mathbf{k}_1=\mu\mathbf{k}/(\mu+\mathbf{m})$, $\mathbf{k}_2=m\mathbf{k}/(\mu+m)$, and $\nabla=\partial/\partial\mathbf{r}$. Wave functions $\Psi_i(\mathbf{r})$ and $\Psi_f(\mathbf{r})$ describe the relative motion of the electron and muon in the initial and final states, respectively. The photon energy ω is related to the initial electron momentum $\mathbf{p}_-\equiv\mathbf{p}$ and the binding energy of the muonium by

$$\omega+\frac{\omega^2}{2(m+\mu-\Delta)}=\Delta+\frac{p^2}{2\mu m}(m+\mu). \tag{4}$$

In the problem under consideration the wave functions Ψ_i and Ψ_f must be eigenfunctions of the same Hamiltonian. Since Ψ_f is the bound-state wave function, it is obvious that the influence of the interaction on the wave function must be taken into account in the initial state, too. This means that $\Psi_i(\mathbf{r})$ has the structure of the wave function of the continuous spectrum of the charged particle in the Coulomb field, and this ensures the orthogonality of $\Psi_i(\mathbf{r})$ and $\Psi_f(\mathbf{r})$. In the process (1) the free electron and muon exist only in the initial state, and therefore the function $\Psi_i(\mathbf{r})$ at large $\mathbf{r}$ must have the form of a superposition of the plane wave and the outgoing spherical wave. A function with this asymptotic form is[9]

$$\Psi_i(\mathbf{r})=\exp(\pi\zeta/2)\Gamma(1-i\zeta)F[i\zeta,1,i(pr-\mathbf{pr})]e^{i\mathbf{pr}}\equiv\Psi(\mathbf{p},\mathbf{r}), \tag{5}$$

where $\zeta=\alpha m\mu/p(m+\mu)$, $\alpha=1/137$ is the fine structure constant, and F is the hypergeometric function. Note that the function $\Psi(\mathbf{p},\mathbf{r})$ is normalized in the same way as the plane wave

$$\int \Psi^*(\mathbf{p},\mathbf{r})\Psi(\mathbf{p}',\mathbf{r})d^3r=(2\pi)^3\delta(\mathbf{p}-\mathbf{p}').$$

We take the ground-state wave function of the muonium as

$$\Psi_f=\sqrt{\frac{\eta^3}{\pi}}e^{-r\eta},\qquad \eta=\frac{m\mu\alpha}{m+\mu}\equiv\zeta p, \tag{6}$$

where $1/\eta$ is the muonium radius.

Using wave functions (5) and (6) and the Gauss theorem we obtain

$$\bar{M}=N\int d^3re^{-r\eta}\frac{\mathbf{ar}}{r}\left(\frac{e^{-i\mathbf{k}_1\mathbf{r}}}{m}+\frac{e^{i\mathbf{k}_2\mathbf{r}}}{\mu}\right)e^{i\mathbf{pr}}F[i\zeta,1,i(pr-\mathbf{pr})], \tag{7}$$

where

$$N=\sqrt{\eta^5/\pi}\exp(\pi\zeta/2)\Gamma(1-i\zeta).$$

The integral in Eq. (7) can be evaluated with the help of the formula[10]

$$\int e^{i(\mathbf{p}-\mathbf{k})\mathbf{r}-r\eta}F[i\zeta,1,i(pr-\mathbf{pr})]d^3r/r=4\pi\frac{[k^2+(\eta-ip)^2]^{-i\zeta}}{[(\mathbf{p}-\mathbf{k})^2+\eta^2]^{1-i\zeta}}.$$

It is easy to show that

$$\bar{M}=i(1-i\zeta)N8\pi(\mathbf{ap})\left[\frac{[\mathbf{k}_1^2+(\eta-ip)^2]^{i-\zeta}}{m[(\mathbf{p}-\mathbf{k}_1)^2+\eta^2]^{2-i\zeta}}+\frac{[\mathbf{k}_2^2+(\eta-ip)^2]^{-i\zeta}}{\mu[(\mathbf{p}+\mathbf{k}_2)^2+\eta^2]^{2-i\zeta}}\right]. \tag{8}$$

When writing the last formula we took into account that $\mathbf{ak}_1=\mathbf{ak}_2=0$.

The square of the matrix element will be defined as

$$|\bar{M}|^2=\frac{2^8\pi^2\zeta(1+\zeta^2)}{1-e^{-2\pi\zeta}}(\mathbf{ap})^2\eta^5A, \tag{9}$$

where

$$A=\frac{e^{-2\zeta\phi_1}}{m^2a_1^4}+\frac{e^{-2\zeta\phi_2}}{\mu^2a_2^4}+\frac{2\cos\left[\zeta\left(\ln\left(\frac{r_2}{r_1}r_{12}\right)\right)\right]}{m\mu a_1^2a_2^2}e^{-(\phi_1+\phi_2)},$$

$$a_1=(\mathbf{p}-\mathbf{k}_1)^2+\eta^2,\quad a_2=(\mathbf{p}+\mathbf{k}_2)^2+\eta^2,\quad \phi_{1,2}=\arctan\frac{2p\eta}{k_{1,2}^2+\eta^2-p^2},$$

$$r_{1,2}=|k_{1,2}^2+(\eta+ip)^2|,\quad r_{12}=\frac{a_1}{a_2}.$$

It follows from (4) that the photon energy $\omega\ll p$ if the kinetic energy of relative motion of the initial particles is greater than or of the same order as the muonium binding energy. Therefore the expression for A can be expanded in this limited case as

$$A=\frac{1}{(p^2+\eta^2)^4}\left\{\frac{(m+\mu)^2}{m^2\mu^2}+\frac{8p\cos\Theta}{p^2+\eta^2}\left(\frac{k_1}{m^2}-\frac{k_2}{\mu^2}+\frac{k_1-k_2}{m\mu}\right)\right.$$
$$+\left[\frac{2}{p^2+\eta^2}\left(-2+\frac{20p^2\cos^2\Theta}{p^2+\eta^2}\right)+\frac{4\eta^2}{(p^2+\eta^2)^2}\right]\left(\frac{k_1^2}{m^2}+\frac{k_2^2}{\mu^2}\right)$$
$$+\frac{2}{m\mu}\left[\frac{k_1^2+k_2^2}{p^2+\eta^2}\left(-2+\frac{12p^2\cos^2\Theta}{p^2+\eta^2}\right)-\frac{16p^2\cos^2\Theta}{(p^2+\eta^2)^2}k_1k_2\right]$$

$$+\frac{2}{m\mu}\left[\frac{2\eta^2(k_1^2+k_2^2)}{(p^2+\eta^2)^2}-\frac{2\eta^2\cos^2\Theta}{(p^2+\eta^2)^3}(k_1^2+k_2^2)^2\right\}, \tag{10}$$

where Θ is the angle between vectors **p** and **k**.

If the momentum of relative motion of the initial particles $p \ll \eta$, the corresponding expression will be given by the formula

$$A=\frac{e^{-4}}{\eta^8}\left[\frac{(m+\mu)^2}{m^2\mu^2}\left(1-\frac{8p^2}{3\eta^2}\right)+\frac{8p\cos\Theta}{\eta^2(m+\mu)}\left(\frac{k_1}{m}-\frac{k_2}{\mu}\right)-\frac{4(k_1+k_2)^2\cos\Theta}{m\mu\eta^2}\right]. \tag{10a}$$

When writing (11) we took into account that in this limiting case the photon energy is of the order of the muonium binding energy: $\omega \cong \Delta \approx \alpha\eta \ll \eta$.

The differential probability (2) defines the differential cross section $d\sigma = dw/v_r$, where $v_r = p(m + \mu)/m\mu$ is the relative velocity of the electron and muon in the initial state. (We take the particle densities equal to unity).

Thus, summing over photon polarizations, we can write the differential cross section of process (1) in the case $p \gg \eta$ as follows [arccot=$\cot^{-1}$]:

$$d\sigma=2^7\pi\frac{\omega}{p}\zeta\left(\frac{\zeta^2}{1+\zeta^2}\right)^3\frac{e^{-4\operatorname{arccot}\zeta}}{1-e^{-2\pi}}d\Omega\,\sin^2\Theta\left\{\frac{(m+\mu)^2}{m^2\mu^2}+\frac{8\omega(\mu^2-m^2)\cos\Theta}{p(1+\zeta^2)m^2\mu^2}\right.$$
$$+\frac{4\omega^2}{p^2(1+\zeta^2)^2m^2\mu^2}[(m^2+\mu^2)(10\cos^2\Theta-1)$$
$$\left.+m\mu(1-\cos^2\Theta(14+\zeta^2))]\right\}. \tag{11}$$

If the initial particles have equal masses, $\mu = m$ (as in the case of radiative transition of an electron and a positron to positronium), the second term in the brackets is absent.

The angular integration of the right-hand side of Eq. (12) gives the following expression for the total cross section for radiative capture of the electron by the muon:

$$\sigma=\frac{2^{10}\pi^2\omega}{3pm^2\mu^2}\left(\frac{\zeta^2}{1+\zeta^2}\right)^3\frac{e^{-4\arctan\zeta}}{(1-e^{-2\pi\zeta})}\left[(m+\mu)^2+\frac{4\omega^2[5(\mu-m)^2+m\mu(1-\zeta^2)]}{5p^2(1+\zeta^2)^2}\right]. \tag{12}$$

Formula (13) is valid for the cross section for radiative capture of oppositely charged particles with arbitrary masses which interact by the Coulomb law. It applies to the process of positronium formation in the collision of a free electron and positron as well as to muonium formation and to the capture of an electron, muon, or τ-lepton by a proton.

2. In the case of the radiative transition of an electron and a positron to positronium we have

$$\sigma=\frac{2^{12}\pi^2\omega}{3pm^2}\zeta\left(\frac{\zeta^2}{1+\zeta^2}\right)^3\frac{e^{-4\arctan\zeta}}{(1-e^{-2\pi\zeta})}\left[1+\frac{\omega^2(1-\zeta^2)}{5p^2}\right],\quad \zeta=\frac{\alpha m}{2p}, \tag{13}$$

and the photon energy is determined from the energy conservation law:

TABLE I. Total cross section for positronium formation.

p (keV/c)	σ (cm^2)	p (keV/c)	σ (cm^2)
1	$3.92\cdot 10^{-21}$	21.5	$4.34\cdot 10^{-26}$
2.15	$5.2\cdot 10^{-22}$	46.4	$1.08\cdot 10^{-27}$
4.46	$3.69\cdot 10^{-23}$	100	$2.41\cdot 10^{-29}$
10	$1.48\cdot 10^{-24}$		

$$\omega+\omega^2/4m=p^2/m+\alpha^2 m/4.$$

This formula is valid if $\zeta\leqslant 1$.

If the relative momentum of the initial particles is very small, and $\zeta\gg 1$, then the differential cross section for positronium formation has the following form:

$$\sigma=\frac{2^7\pi^2\alpha^3 e^{-4}}{p^2}\sin^2\Theta\left[1-\frac{2p^2+12\omega^2\cos^2\Theta}{3\eta^2}\right]d\cos\Theta. \tag{14}$$

The angular integration of (14) gives the total cross section

$$\sigma=\frac{2^9\pi^2\alpha^3 e^{-4}}{3p^2}\left[1-\frac{10p^2+12\omega^2}{15\eta^2}\right].$$

We see that in the case of small initial electron momenta the cross section is proportional to p^{-2}, in contrast to the cross section for slow neutron capture by a proton, which in this case behaves as p^{-1}. This is a consequence of the long-range Coulomb interaction as compared with the short-range strong interaction.

Note that the total cross section σ for positronium formation was derived without taking the particle spins into account. Consequently, in order to obtain singlet positronium production we must multiply it by 1/4. In the case of triplet positronium we have to sum over spin states of the positronium, and this gives a trebled value of the cross section:

$$\sigma^{\uparrow\downarrow}=\frac{1}{4}\sigma,\quad \sigma^{\uparrow\uparrow}=\frac{3}{4}\sigma.$$

Table I lists the total cross sections calculated according to formula (14) for the radiative transition of an electron and positron to positronium at p values ranging between 1 keV/c and 100 keV/c.

The cross section σ increases and reaches very large values as the energy of the initial particles decreases. In consequence, the process considered should be taken into account when investigating the properties of electron–positron plasmas and the low-energy annihilation of electrons and positrons into photons.

The authors thank S. V. Peletminsky for helpful discussions and critical remarks.

This work was supported in part by the International Soros Science Education Program (ISSEP).

[1)]e-mail: kfti@rocket.kharkov.ua

[1]J. Pirenne, Arch. Sci. Phys. Nat. **29**, 207 (1947); V. B. Berestetsky and L. D. Landau, Zh. Éksp. Teor. Fiz. **19**, 673 (1949); V. B. Berestetsky, Zh. Éksp. Teor. Fiz. **19**, 1130 (1949).
[2]I. Pomeranchuk, DAN SSSR **60**, 218 (1947); A. Ore and J. L. Powell, Phys. Rev. **75**, 1696 (1949).
[3]J. C. Cheng and P. J. Kramer, Phys. Rev. A **5**, 1207 (1972).
[4]N. K. Sarkar, M. Basu, and A. S. Ghosh, Phys. Rev. A **45**, 6887 (1992); M. Basu and A. S. Ghosh, J. Phys. B **21**, 3439 (1988).
[5]A. Jagarashi and N. Toshima, Phys. Rev. A **46**, R1159 (1992); N. C. Deb, J. H. McGuire, and N. C. Sil, Phys. Rev. A **36**, 1082 3707 (1987).
[6]T. D. Steiger and R. S. Conti, Phys. Rev. A **45**, 2744 (1992); S. Tripathi and C. Sinha, Phys. Rev. A **42**, 5743 (1990).
[7]E. Fermi, Phys. Rev. **48**, 570 (1935).
[8]A. I. Akhiezer, S. V. Peletminsky, and A. A. Yatsenko, Low-Temp. Phys. **20**, 509 (1994).
[9]A. Sommerfeld, *Atombau und Spektrallinien*, II. Band. Frieder. Vieweg & Sohn, Braunschweig (1951).
[10]A. I. Akhiezer and V. B. Berestetsky, *Quantum Electrodynamics* [in Russian], Moscow Nauka (1969).

Published in English in the original Russian journal. Edited by Steve Torstveit.