Equations of a relativistic gravitational theory embedded in a space with a dimensionality greater than four

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(Submitted 10 April 1987)
Pis'ma Zh. Eksp. Teor. Fiz. **46**, No. 1, 3–5 (10 July 1987)

A generalization of the equations of a relativistic gravitational theory to a space with a dimensionality $D \geqslant 4$ is proposed.

The relativistic gravitational theory formulated in Ref. 1 and developed in Ref. 2 is based on Minkowski space and the principle of geometrization. In other words, the initial space of the relativistic gravitational theory is (M_4,γ): a smooth, oriented manifold M_4, which can be described by means of a single map provided by a pseudo-Euclidean metric γ. For each physical motion of the matter under the influence of the gravitational field in (M_4,γ) there exists a single-valued transform in the effective space (M_4^*,g), constructed in the relativistic gravitational theory by means of the complete system of equations of the theory. Here M_4^* is a trivial single-layer covering of the initial manifold, provided by the pseudo-Riemannian metric g. The complete system of equations of the relativistic gravitational theory is

$$R_{ij} - g_{ij}R/2 = 8\pi T_{ij}\,; \quad D_j\tilde{g}^{ij} = 0;$$

where D_i is a covariant derivative in (M_4,γ), and $\tilde{g}^{ij} = \sqrt{-g}\, g^{ij}$.

In a sense, this theory is a development of the ideas of Ref. 3. We wish to propose the following generalization of the equations of this theory. Instead of the manifold M_4, we consider the manifold M_D of dimensionality $D \geqslant 4$: $M_D = M_4 \times K$, where M_4 is a Minkowski space with the metric γ, and K is an "auxiliary" manifold. We denote the coordinates in M_D by Q^A, $A = 0,1,...,D-1$; if $A = i = 0,...,3$, then we have $Q^i \in M_4$. If $A = N = 4,...,D-1$, then we have $Q^N \in K$. We assume that the metric in M_D is γ_{AB}, while the coordinates in M_4^* are x^μ, $\mu = 0,...,3$.

The nature of manifold K may be extremely varied: For example, K could consist of "compact" spaces in the sense of Ref. 4 or various "inner" spaces in the sense of Ref. 5; alternatively, $M_4 \times K$ might describe a "continuum" of "copies" of space M_4.

The coupling (embedding) of the two spaces (M_4^*,g) and (M_D,γ) is described by the equation of a surface, $Q^A = Q^A(x)$, with an induced pseudo-Riemannian metric $g_{\mu\nu}(x)$. This equation can be extracted from the theory of minimal surfaces.[6,7] For this purpose, we take the action S in the form

$$S = \int d^4x \,\{\tilde{g}^{\mu\nu}(x)\,\partial_\mu Q^A\,\partial_\nu Q^B \gamma_{AB}\frac{1}{\alpha} + \sqrt{-g(x)}\,R(x)\,\frac{1}{k} - \sqrt{-g(x)}\,\frac{2}{\alpha} + L\,(g_{\mu\nu}(x),\,[Q^A])\,\}\,. \quad (1)$$

Here α, k are coupling parameters; L is the Lagrangian of the "matter"; the first term in (1) is standard for descriptions of minimal surfaces; the second stems from the induced curvature on the surface $Q^A = Q^A(x)$; and the third provides a solution of the theory in the form of a plane space $(M_4^*, g) = (M_4, \gamma)$ in the absence of "matter."

The Euler-Lagrange equations $\partial S/\delta Q^A = 0$ and $\delta S/\delta g^{\mu\nu} = 0$ for action (1) give us

$$\partial_\mu (\tilde{g}^{\mu\nu}\partial_\nu Q^A) = \frac{\alpha}{2}\gamma^{BA}\frac{\delta L}{\delta Q^B} + \frac{1}{2}\gamma^{BA}\tilde{g}^{\mu\nu}\partial_\mu Q^C \partial_\nu Q^D \frac{\partial\gamma_{CD}}{\partial Q^B}$$

$$-\tilde{g}^{\mu\nu}\partial_\mu Q^B \gamma^{AC}\partial_\nu \gamma_{BC}, \quad (2)$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\frac{k}{\sqrt{-g}}\frac{\delta L}{\delta g^{\mu\nu}} - \frac{k}{\alpha}\gamma_{AB}(\partial_\mu Q^A \partial_\nu Q^B - \frac{1}{2}g_{\mu\nu}g^{\sigma\tau}$$

$$\times \partial_\sigma Q^A \partial_\tau Q^B) - \frac{k}{\alpha}g_{\mu\nu}.$$

Equations (2) generalize the relativistic gravitational theory to a space with dimensionality $D \geqslant 4$. Equations (2) simplify substantially in the first ("background") approximation:

$$Q^A = Q^A(x) = \left| \begin{array}{lll} \xi^i(x) & \in M_4, & \\ \epsilon\eta^N, & \epsilon \to 0, & \in K. \end{array} \right. \quad (3)$$

After some manipulations involving (3) and the rules for making a tensor transformation from the variables x^μ to the variables ξ^i, these equations become

$$\partial_\mu (\tilde{g}^{\mu\nu}\partial_\nu \xi^i) = -\gamma^i_{jk}\partial_\mu \xi^j \partial_\nu \xi^k \tilde{g}^{\mu\nu}; \quad (4)$$

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi T_{ij} - \frac{k}{\alpha}(\gamma_{ij} + g_{ij} - \frac{1}{2}g_{ij}g^{mn}\gamma_{mn}). \quad (5)$$

Here the γ^i_{jk} are the Christoffel symbols in the metric $\gamma_{ij}(\xi) \in M_4$, and $T_{ij} = -k\delta L/\delta g^{ij} 8\pi\sqrt{-g}$. Equation (4) is the same[2] as the equation $D_\mu \tilde{g}^{\mu\nu} = 0$ of the relativistic gravitational theory; Eq. (5) is an equation of a relativistic gravitational theory with a "cosmological term"[8] and is also reminiscent of the Hilbert-Einstein equation with a specially selected "mass" term.[9] As in Refs. 8 and 9, the Bianchi identity for (5) reduces to an equation for the "matter," $\nabla_i T^{ij} = 0$, and to Eq. (4) in the form $D_\mu \tilde{g}^{\mu\nu} = 0$. The requirement $k/\alpha > 0$ in (5) leads to the appearance of a mass of the gravitons; this mass, which is proportional to $\sqrt{k/\alpha}$, causes major changes in the picture of the evolution of the universe even in the case of the simplest cosmological model: The minimum value of the scale factor for a homogeneous and isotropic universe is not zero and is instead proportional to $\sqrt{k/\alpha}$, the mass of the graviton. At small values of k/α, all the other properties of the Big Bang remain the same. A sufficiently small value of k/α also would not be manifested in the gravitational effects observable at present.

In the following approximations, the solutions of system (2) depend on the nature of manifold K; in general, this situation violates the standard principle of equivalence.

I wish to thank A. A. Logunov for many valuable discussions and also A. P. Isaev, F. A. Lunev, and K. A. Sveshnikov.

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Translated by Dave Parsons