State density of fractal structures with a long-range interaction

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The example of a triangular Serpinskiĭ carpet with an $r^{-\beta}$ interaction between nodes is used to show that for values of β above a critical value β_c the state density is determined by a universal exponent (the spectral dimensionality), as in the case of a short-range interaction. At $\beta < \beta_c$, however, the exponent depends on β. This exponent is found through an ϵ expansion around β_c.

Let us consider the equation

$$E\psi_i = -\ \Sigma V_{ij}\psi_j, \tag{1}$$

where the wave functions are defined at the nodes of a structure which has a scaling symmetry (a fractal).[1] As has been established for percolation systems[2,3] and also for a wide class of ordered fractal structures,[3–5] the density of low-frequency states (near the bottom of the band, E_b) in the case of a short-range interaction V_{ij} is determined by a universal exponent or spectral dimensionality d_s:$\rho(E) \sim (E - E_b)^{\mu}$, where $\mu = (d_s/2) - 1$. The same is true for systems having translational symmetry, where d_s is the dimensionality of the space. On the other hand, we know that in the latter case a long-range interaction leads to a change in μ. In particular, in a two-dimensional system ($d_s = 2$) we would have $\mu = 0$ for a short-range interaction, while in (for example) a two-dimensional lattice of dipole oscillations oriented normal to the plane ($V_{ij} \sim r_{ij}^{-3}$) we would have $\mu = 1$. The reason for this behavior is a change in the dispersion law from $E - E_b \sim k^2$ (k is the wave vector) in the case of a short-range interaction to $E - E_b \sim k$ with $V_{ij} \sim r_{ij}^{-3}$. A question which naturally arises is that of the role of a long-range interaction in the low-frequency behavior $\rho(E)$ of fractal structures. This question is pertinent to a long list of problems, in particular, the absorption of light in large clusters of small metal particles at frequencies near the frequencies of the natural dipole plasma oscillations of these particles and the exciton states of macromolecules. The same situation arises in the problem of energy transport with a power-law dependence of the excitation hopping time on the distance between sites.

In this letter we consider scalar equation (1) defined at the nodes of a triangular Serpinskiĭ carpet (Fig. 1). The interaction V_{ij} is $V_{ij} = \alpha_{ij} + \gamma_{ij}$, where $\alpha_{ij} = 1$, when i and j belong to nearest nodes or 0 in all other cases, and $\gamma_{ij} = \gamma|\mathbf{r}_i - \mathbf{r}_j|^{-\beta}$. As in the case[3–5] $\gamma_{ij} = 0$, we use a recursion procedure which involves elminating the variables ψ_i at interior nodes (Fig. 1) with a renormalization of the parameters of Eq. (1). We use a perturbation theory in γ. It turns out that at $\beta < \beta_c$, there are two fixed points. One is the trivial point $\gamma^* = 0$, which corresponds to a short-range interaction, and the

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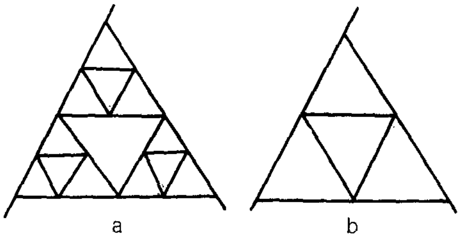

FIG. 1.

other is a nontrivial point at which we have $\gamma^* \neq 0$. Under the condition $\beta = \beta_c - \epsilon (\epsilon \ll 1)$; we have $\gamma^* \sim \epsilon$, justifying the use of a perturbation theory in γ. The picture is analogous to the ϵ expansion in the theory of phase transitions.[6]

Upon renormalization, γ_{ij} does not retain a power-law form over distances $r_{ij} \sim 1$. At large distances, however, the power-law dependence is retained, within an error $\sim r_{ij}^{-1}$. In this region, near a fixed point ($\gamma_{ij} = 0, E = -4$), we have $\gamma'_{ij} = 15\gamma_{ij}$ for the renormalized values. Substituting in $\gamma_{ij} = \gamma r_{ij}^{-\beta}$, we find $\gamma' = \gamma 15/2^{\beta}$. It follows that under the condition $\beta > \beta_c = \ln 15/\ln 2$ the constant γ tends toward the fixed point $\gamma^* = 0$. The system thus behaves as it would in the case of a short-range interaction. Since we have $d_s = 2\ln 3/\ln 5$ for a two-dimensional Serpinskiĭ carpet, and the fractal dimensionality is[1] $d_f = \ln 3/\ln 2$, we find

$$\beta_c = d_f + 2\frac{d_f}{d_s}. \tag{2}$$

It is a straightforward matter to derive the same relation for an n-dimensional Serpinskiĭ carpet. It follows from simple considerations that this relation also holds for fractal structures of a general type, including disordered structures. To demonstrate the point, we note that in a system with translational symmetry the exponent β_c is determined by the convergence of the sum $\Sigma r_{ij}^{2-\beta}$, which arises when the right side of (1) is expanded in small gradients of ψ_i. In the case of fractal structures, an expansion in gradients is not so trivial a matter, since ψ_i oscillates at arbitrarily small scales. We can make use of the concept of a fractal derivative.[7] In this case, the sum becomes $\Sigma\, r_{ij}^{\nu-\beta}$. The exponent of the fractal derivative, ν, is unambiguously related to[2] d_s; relation (2) then follows.

For $\beta < \beta_c$ the point $\gamma^* = 0$ is unstable. The recursion relations in the case $\gamma \ll 1$ are

$$W' = 5\,W(1 - a\gamma), \qquad \gamma' = \frac{15}{2^{\beta}}\,\gamma(1 - a\gamma), \tag{3}$$

where $W = E + 4 + (1/N)\Sigma_{ij}\gamma_{ij}$ (N is the number of nodes). The equation $W = 0$ determines a lower limit on the eigenstate spectrum in the approximation linear in γ_{ij}. Relations (3) hold near the point $W = 0$. The numerical parameter a is ~ 1. It is determined by the short-range interaction, because if the rapid convergence of sums of the type γ_{ij}. When the interactions in the region $|\mathbf{r}_i - \mathbf{r}_j| \leqslant 2\sqrt{3}$, are taken into account (this approach corresponds to the nearest long-range neighbors in lattice b in Fig. 1), we have $a \approx 0.4$. At these distances, the quantities γ_{ij} are of course not power-law functions. Their values at a fixed point were found from an analysis of the complete system of recursion relations in this nearby region. It follows from (3) that at the nontrivial fixed point $W^* = 0$ we also have $\gamma^* = \epsilon \ln 2/a$, where $\epsilon = \beta_c - \beta \ll 1$. The state density is found from the equation $\rho(W')dW' = 2^d f \rho(W) dW$. Hence, using (3), we find $\rho \sim W^{(\tilde{d}_s/2) - 1}$, where the effective spectral dimensionality is

$$\tilde{d}_s = d_s \left(1 - \frac{d_s}{2d_f}\epsilon\right)^{-1}. \quad (4)$$

It should be noted that for an arbitrary fractal structure the same equation could be derived directly from (1) by assuming that the ψ_i are characterized by a hierarchy of scales. These scales increase up to the largest scale which is, in some sense, an analog of the wavelength in a translationally symmetric system.

The long-range effect has been studied previously on Koch lines in a hierarchical model (nearest neighbors interact at each level of the hierarchy).[8] It is easy to show that a model of this sort has an exact solution on a triangular Serpinskiĭ carpet, as in Ref. 8. There is, however, no second universal exponent of spectral dimensionality.

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