Euler instability in discotic liquid crystals

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Stretching or contracting a discotic liquid crystal causes it to become unstable, which leads to a deformation of the discotic filaments. The instability threshold and a simple modulation period are found.

The phenomenon of elastic instability of a solid rod, which is subjected to the action of a compressive force (the Euler instability), can be described as one in which the linear shape of the rod corresponds to an unstable equilibrium when a certain critical value of the compressive force is reached. An infinitesimal compressive force is all that is necessary to disrupt the equilibrium and to bend the rod.

Layered (cholesteric and smectic) liquid crystals reveal a similar phenomenon which occurs when they are stretched in the direction perpendicular to their layers.[1] As a result of being stretched to a certain critical value, the flat layers become unstable, and the so-called wavelike or fluctuating modulation commences. Although the formal description of this instability, given by Helfrich,[2] is similar to the description of Euler's instability in solid rods, the physical causes of these phenomena are completely different. The fluctuating modulation in cholesteric and smectic liquid crystals actually occurs because of the absence of the modulus for the relative displacement of the layers (i.e., because of the softness of the correlation function for the displacement of the levels). The manner in which each instability manifests itself is also different. The Euler rod instability, for example, corresponds to a one-dimensional modulation, whereas the Helfrich instability gives rise to a two-dimensional layer modulation and to the formation of a square grid.[2]

The wavelike instability in smectic and cholesteric liquid crystals, which has now been studied extensively, has found many applications.[3]

The physical causes of the instability (the softness of the correlation function of the displacements) exist, however, even in a more solid class of liquid crystals: discotics whose structure is described by a two-dimensional lattice of liquid-crystal filaments. We will show below that stretching this lattice beyond a certain threshold causes a one-dimensional modulation of the filaments. The instability of the discotic filaments in this sense resembles Euler's instability more closely than does the fluctuating modulation in smectic and cholesteric liquid crystals. The instability of straight liquid-crystal strands in discotics, like the fluctuating modulation, may be caused by the magnetic field.

Let us consider the static stability of straight filaments of the most common modification of hexagonal discotics. The elastic energy of discotics can easily be described by two scalar functions, W_1 and W_2, which are the density modulation phases in two directions.[4] By virtue of their definition, the functions W_1 and W_2 are such that

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a system of two equations $W_{1,2}(\mathbf{r},t) = \text{const}$ specifies the position in space and the time evolution of a certain discotic liquid-crystal filament. Because of this property, the unit vector $\vec{\nu} = [\nabla W_1 \times \nabla W_2]/|[\nabla W_1 \times \nabla W_2]|$ is directed along the filament.

The expansion of the energy in gradients of W_i must (a) be rotationally invariant and (b) be invariant with respect to the symmetry elements of the lattice (we should bear in mind here that the lattice rotation does not affect the index of W_i). In hexagonal discotics these rules define the following form of the energy expansion within fourth-order terms in ∇W_i:

$$E_W = \frac{B_t - B_l}{4q^2}(\nabla W_i)^2 + \frac{(B_l - 2B_t)}{8q^4}(\nabla W_i \nabla W_i)^2 + \frac{B_t}{4q^4}(\nabla W_i \nabla W_j)^2 + \frac{K}{2q^2}(\nabla^2 W_i)^2, \tag{1}$$

where B_t and B_l are the transverse and longitudinal elastic moduli of the lattice of discotic liquid-crystal filaments, q is the wave vector of the hexagonal discotic density modulation, and K is the Frank modulus.

The minimum of energy (1) is found from the solution

$$W_{10} = qx, \qquad W_{20} = qy. \tag{2}$$

The two equations, $W_1 = \text{const}$ and $W_2 = \text{const}$ with W_i in (2), describe the system of liquid-crystal filaments that are parallel to the z axis. The vector $\vec{\nu}$ is also directed along this axis. Deformation of the system of liquid-crystal filaments causes the functions W_i to be different from those in (2). To describe this deformation, we use the notation

$$W_1 = q(x - U_x), \qquad W_2 = q(y - U_y). \tag{3}$$

In the case of small deviations from equilibrium, the vector U_α coincides with the displacement of the filaments along the x and y axes. Substituting (3) into (1), we find the harmonic elastic energy of the discotic, $E_W^{(0)}$, and the dominant third-order anharmonic terms $E_W^{(1)}$

$$E_W^{(0)} = \frac{B_l}{2}(\nabla_\alpha U_\alpha)^2 + \frac{B_t}{2}(\epsilon_{\alpha\beta}\nabla_\alpha U_\beta)^2 + \frac{K}{2}(\nabla^2 U_\alpha)^2, \tag{4}$$

$$E_W^{(1)} = -\frac{(B_l - 2B_t)}{2}\nabla_\alpha U_\alpha (\nabla U_\beta)^2 - B_t \nabla_\alpha U_\beta \nabla U_\alpha \nabla U_\beta, \tag{5}$$

where $\epsilon_{\alpha\beta}$ is a two-dimensional antisymmetric tensor.

Let us now test (4) and (5) with respect to the stability against a simple deformation:

$$U_\alpha = \gamma x_\alpha + u_\alpha \cos k_z z \sin k_x x \sin k_y y. \tag{6}$$

The parameter γ corresponds to the stretching of the filament grid, u_α is the perturbation of the straight filaments, $k_z \equiv k$ is the wave vector of this perturbation, and the wave vectors k_x and k_y are determined by the size of the sample and by the boundary conditions. In the simple case of a square sample, cut along the x and y axes, of size L, we find from the conditions $u_\alpha(0) = u_\alpha(L) = 0$ the relation $k_x = k_y = \pi/L = q_0$. Substituting (6) into (4) and (5), we find the following expression for the terms which are quadratic in u_α in the elastic energy

$$E = \frac{B_l q_0^2}{2}(u_x + u_y)^2 + \frac{B_t q_0^2}{2}(u_x - u_y)^2 + \frac{Kk^4}{2}(u_x^2 + u_y^2) - \frac{\gamma k^2}{2}(B_l - 4B_t)(u_x^2 + u_y^2). \tag{7}$$

In principle, there can be different types of discotic-filament perturbations. It is easy to see, however, that the first instability is symmetric (i.e., $u_x = u_y$). The straight filaments in this case become unstable when

$$\gamma \geqslant \gamma_c = \frac{(B_l + B_t)q_0^2 + Kk^4}{(B_l - 4B_t)k^2}. \tag{8}$$

An important difference between this result and that for the smectic and cholesteric liquid crystals is that in discotics the straight filaments may become unstable as a result of stretching (if $B_l > 4B_t$) or as a result of contraction (if $B_l < 4B_t$).

The modulation period of the filament near the instability threshold can be found, as usual, by minimizing (8) with respect to

$$k_c = [(B_l + B_t)q_0^2/K]^{1/2}. \tag{9}$$

We conclude with the remark that the instability that arises and the manner in which it develops depend strongly on the type of the initial perturbation and the shape of the sample. These topics and the dynamics of the instability development will be discussed in a separate paper.

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