Zeros in the fermion spectrum in superfluid systems as diabolical points

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The gap in the energy spectrum of fermion excitations in superfluid Fermi systems may vanish at points or lines on the Fermi surface. The only zeros that are stable with respect to external excitations, however, are those which are diabolical points (or lines) of the spectrum and which therefore have a topological charge.

If the gap in the spectrum $E(\mathbf{k})$ of fermion excitations ($\mathbf{k}$ is the momentum or quasimomentum of the fermions) vanishes at points or lines in the space of the momentum $\mathbf{k}$ of a superfluid Fermi liquid, such a liquid assumes an intermediate position between "classical" superfluid systems, in which there are no zeros in the gap, and a normal Fermi liquid. In such a system the normal motion of low-energy fermions is important even at $T = 0$ and gives rise to several features. In superfluid ^{3}He-A, for example, where the gap vanishes at two points on the Fermi surface, the normal fermions are chiral fermions, and their interaction with the order parameter gives rise to phenomena known in quantum field theory as the chiral anomaly.[1] It has been suggested that there are also zeros in the gap in certain superconductors belonging to heavy-fermion systems (see the review by Lee *et al.*[2]). The existence of zeros is a consequence of a nontrival type of Cooper pairing, in which gauge invariance is not completely disrupted.[3] The existence of zeros and their type are dictated by the symmetry of the ordered state, i.e., by the class of superfluidity or superconductivity.[3,4]

An important point is the stability of the zeros with respect to external perturbations which disrupt the symmetry of the coherent state. While some of the zeros vanish when subjected to arbitrarily small perturbations (the low-temperature thermodynamics of the system is disturbed), others survive in the face of any perturbations that are moderately strong. For example, only some of the zeros survive in systems with heavy fermions if (for example) an elastic stress is applied or if we go beyond the scope of the single-band approximation and incorporate scattering processes, while the stability of the point zeros in ^{3}He-A is provided by the topology of the order parameter $\Delta_{ss'}(\mathbf{k})$ in momentum space[5] (the subscript "s" in ^{3}He specifies the spin state). In the more general case of heavy fermions, in which the crystal field is important and the subscript "s" specifies not only the spin state but also the band, however, the theory of the topological stability of the zeros derived in Ref. 5 is not applicable. This theory can be generalized by making use of the general properties for a crossing of branches of a spectrum which were found by Von Neumann and Wigner[6] and which have once again attracted interest because of the appearance of the Berry phase[7]: Specifically, the only zeros in a spectrum that are stable are those which are "diabolical points" of the spectrum.

Diabolical points (see also Ref. 8, where they were studied in nuclear physics) are exceptional points in the spectrum in the sense that two different branches of the spectrum touch at these points. Let us examine the general perturbations which do not conserve the symmetry, so that the branches of the spectrum do not differ in symmetry and thus cannot cross. The only possibility is a touching of branches, and then only if the dimensionality of the space of the parameters on which the spectrum depends is[6] $\geqslant 3$. Since the dimensionality of the $\mathbf{k}$ momentum space is 3, the two branches $E_i(\mathbf{k})$ and $E_j(\mathbf{k})$ may touch at points in momentum space. The point at which they touch is topologically stable: Under the influence of external perturbations, it can only move in momentum space or annihilate when it meets a point having the opposite topological charge. The topological charge is determined in the following way.[13]

The spectrum of fermions consists of the eigenvalues of the $N \times N$ Hermitian matrix $H_{ij}(\mathbf{k})$, which depends on k as a parameter, where N is the number of bands, including the spin variables. In addition, the "isospin" in Bogolyubov-Nambu space, i.e., particle and hole states, is taken into account. In the special case of superfluid ^{3}He we have Ne = 4 and

$$H_{ij} = \begin{pmatrix} \epsilon_{ss'}(\mathbf{k}) & \Delta_{ss'}(\mathbf{k}) \\ \Delta^{+}_{ss'}(\mathbf{k}) & -\epsilon^{*}_{ss'}(-\mathbf{k}) \end{pmatrix} \tag{1}$$

which corresponds to four spectral branches [if $\epsilon_{ss'} = \epsilon(\mathbf{k})\delta_{ss'}$ is diagonal and real, we would have $E = \pm\sqrt{\epsilon^2 + |\Delta_1|^2}$, $E = \pm\sqrt{\epsilon^2 + |\Delta_1|^2}$].

The orthonormal set of eigenfunctions $\psi_{ia}(\mathbf{k})$ of the matrix H_{ij} (a specifies the function, while i specifies the components of the function in the column) forms a unitary matrix which diagonalizes H_{ij}:

$$H_{ij}(\mathbf{k}) = \sum_a \psi^{+}_{ja}(\mathbf{k}) E_a(\mathbf{k})\, \psi_{ia}(\mathbf{k})\,. \tag{2}$$

This set maps the $\mathbf{k}$ space into the space of the unitary matrices $U(N)$; the multiplication of $\psi_{ia}(\mathbf{k})$ by phase factors $\exp[i\varphi_a(\mathbf{k})]$ does not change (2). This situation corresponds to a factorization of $U(N)$ with respect to $U^N(1)$. Consequently, the range (R) of the eigenfunctions ψ_{ia} is the factor space $R = U(N)/U^N(1)$, which has a nontrivial homotopic group $\pi_2(R) = Z^{N-1}$. The possible singular points of the mapping of the **k** parameter space into R are thus characterized by $N-1$ integer charges. However, it is convenient to introduce N charges N_a with the condition $\Sigma_a N_a = 0$.

For the general position, only two charges, e.g., N_1 and N_2, are nonzero and take on the values 1 and -1, since the singularities with approximately equal charges split into these very simple elements upon very slight perturbations. This simplest zero corresponds to the contact of two branches of the spectrum at the point $\mathbf{k}_0$ where the mapping has a singularity: $E_1(\mathbf{k}_0) = E_2(\mathbf{k}_0)$. Since the topological charge is conserved under very slight perturbations, the point $\mathbf{k}_0$ can only shift; it cannot disappear. The topological invariants for points at which the branches of a spectrum merge were first introduced by Novikov.[12]

The diabolical points of primary interest for superfluid and superconducting states are those at which the spectra of particles and holes make contact, i.e., where positive and negative levels, e.g., $E_1 < 0$ and $E_2 > 0$, make contact. At such diabolical points lying on the Fermi surface, the spectrum of fermion quasiparticles must vanish. Near such zeros, as in the general case in which two arbitrary branches touch,[7] the fermions are described by a two-level Hamiltonian. This 2×2 Hamiltonian, which is linear in $\mathbf{k}-\mathbf{k}_0$, is given in general in terms of the two-row Pauli matrices σ_α as follows:

$$H = \vec{\sigma}\cdot \mathbf{m}(\mathbf{k}) , \qquad m_\alpha(\mathbf{k}) = e^i_\alpha (k_i - k_{0i}) \tag{3}$$

where the coefficients e^i_α, depend on the particular features of the spectrum. This situation corresponds to the Hamiltonian of a massless chiral fermion which is moving n the field of vector electromagnetic potential $\mathbf{A} = \mathbf{K}_0$ and the gravitational field of triads e^i_α ($\mathbf{A}$ and e^i_α depend on the spatial coodinates $\mathbf{r}$ if the external perturbation is spatially nonuniform).

Like the other diabolical points, the point $\mathbf{k}_0$ in (3) is a "magnetic" monopole in k space.[7] As we circumvent a contour C in $\mathbf{k}$ space, any solution of Eq. (3) acquires a geometric phase factor $\gamma(C)$, called the "Berry phase," which is expressed in terms of an integral of the "magnetic" field $\mathbf{H}(\mathbf{k})$ over a surface resting on this closed contour:

$$\gamma(C) = \iint_C dS\cdot \mathbf{H}(\mathbf{k}) , \qquad H_i(\mathbf{k}) = \frac{1}{4|\mathbf{m}|^3} e_{ijk}\, \mathbf{m}\cdot \left[\frac{\partial \mathbf{m}}{\partial k_j} , \frac{\partial \mathbf{m}}{\partial k_k}\right] . \tag{4}$$

At the point $\mathbf{k} = \mathbf{k}_0$ the field $\mathbf{H}(\mathbf{k})$ has a "magnetic" pole

$$\frac{\partial}{\partial \mathbf{k}} \cdot \mathbf{H}(\mathbf{k}) = 2\pi\delta(\mathbf{k}-\mathbf{k}_0) . \tag{5}$$

Because of this monopole, the eigenfunctions of Hamiltonian (3) cannot be determined globally for all **k**: In any determination of these eigenfunctions, one always finds a Dirac string—a vortex line emerging from the monopole, on which this solution is not defined—for each of the solutions. The reason is that as we traverse an infinitely small contour C around this line, the solution acquires a Berry phase $\gamma(C) = 2\pi$.

The result is the appearance of a chiral anomaly in superfluid ^{3}He-A and, possibly, in certain heavy-fermion systems. [The vortex singularity in the phase $\Phi(\mathbf{k})$ of the Bogolyubov-Nambu functions $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$, which make up the eigenfunction of Eq. (3) in ^{3}He-A, was pointed out some time ago by Leggett and Takagi.[9] These vortex lines in a field Φ have been studied in detail. It has been pointed that the physical results depend not on the positions of the lines but on the position of the monopole from which they emerge—which was called a "boojum" on the Fermi surface in Ref. 5. This is a logical approach, since the choice of a Dirac string is arbitrary and depends on the choice of gauge.]

Zeros which are diabolical points are the only zeros that survive the effect of arbitrary perturbations. Their existence is accompanied by a ferromagnetism or antiferromagnetism[3]; in ^{3}He-A, for example, an orbital ferromagnetism is associated with them. In addition to the points of tangency of branches in a three-dimensional space (a tangency of codimensionality 3), tangencies of codimensionality $n \neq 3$, determined by the group $\pi_{n-1}(R)$, are possible in a certain symmetry. Lines of a tangency of branches in a three-dimensional space ($n = 2$) arises, for example, in the case of real matrices H_{ij} (Refs. 6, 7, and 10). In this case we have $R = SO(N)$ and a nontrivial $\pi_1(R)$ for all N. If branches with $E < 0$ and with $E > 0$ touch, then the gap in the spectrum necessarily vanishes on these lines. Lines of this sort may exist in heavy-fermion systems if external perturbations do not disrupt the required symmetry. Zeros appear in the spectrum of fermions of codimensionality $n = 4$ (a "diabolical instanton") at, for example, a domain wall connecting two different vacuums in ^{3}He-B: The energy of the quasiparticles vanishes at the point $(\mathbf{k},x)$ in the four-dimensional space, where x is the coordinate along the normal to the wall.[11] Zeros of higher codimensionality are possible in topological entities having the shape of lines ($n = 5$) or points ($n = 6$).

In summary, among the zeros of the gap which were found in Ref. 3 in various superconductivity classes, the only ones that survive when umklapp processes are taken into account are those which correspond to diabolical lines and points lying on the Fermi surface. The others acquire a gap, although it is small in terms of the parameter T_c/ϵ_F. If, either spontaneously or under the influence of an external force, a diabolical point or line moves away from the $E = 0$ surface, i.e., away from the Fermi surface, then one of the branches will necessarily intersect the Fermi surface, giving rise to a finite state density and thus to a heat capacity which is linear in T at extremely low temperatures. This is what we see in ^{3}He-A in the presence of a superfluid velocity or if $\text{curl}\mathbf{A} \equiv \text{curl}\mathbf{k}_0$ (see Ref. 1 and the bibliography there).

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[1]G. E. Volovik, Zh. Eksp. Teor. Fiz. **92**, 2116 (1987) [Sov. Phys. JETP **65** (1987) (to be published)]; J. Low Temp. Phys. **67**, 301 (1987).
[2]P. A. Lee, T. M. Rice, J. W. Serene, L. J. Sham, and J. W. Wilkins, Comments Cond. Matt. Phys. **12**, 99 (1986).
[3]G. E. Volovik and L. P. Gor'kov, Zh. Eksp. Teor. Fiz. **88**, 1412 (1985) [Sov. Phys. JETP **61**, 843 (1985)].
[4]E. I. Blount, Phys. Rev. B **32**, 2935 (1985).
[5]G. E. Volovik and V. P. Mineev, Zh. Eksp. Teor. Fiz. **83**, 1025 (1982) [Sov. Phys. JETP **56**, 579 (1982)].
[6]J. Von Neumann and E. P. Wigner, Phys. Z. **30**, 467 (1929).
[7]M. V. Berry, Proc. R. Soc. London **A392**, 45 (1984).
[8]R. S. Nikam and P. Ring, Phys. Rev. Lett. **58**, 980 (1987); R. S. Nikam, P. Ring, and L. F. Canto, Phys. Lett. **B185**, 269 (1987).

[9]A. J. Leggett and S. Takagi, Ann. Phys. 110 353 (1978).
[10]V. I. Arnol'd, Matematicheskie metody klassicheskoĭ mekhaniki (Mathematical Methods of Classical Mechanics), Nauka, Moscow, 1979, Supplement 10.
[11]M. M. Salomaa and G. E. Volovik, " 'Cosmic' domain walls in superfluid ^{3}He-B: instantons in (k,r)-space," Preprint of Helsinki University of Technology, TKK-F-A610.
[12]S. P. Novikov, Dokl. Akad. Nauk SSSR **257**, 538 (1981).
[13]J. E. Avron, R. Seiler, and B. Simon, Phys. Rev. Lett. **51**, 51 (1983).

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