Partition functions in superstring theory. Type 2

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It is proved that the two-loop contribution to the partition function vanishes after a summation over spinor structures.

The vanishing of the partition functions of heterotic and supersymmetric strings is a necessary condition for ten-dimensional supersymmetry and if these theories are to remain finite in the perturbation-expansion approach. Unfortunately, attempts to prove this crucial property have run into considerable difficulties and have not yet succeeded. [The discussions of an extremely general nature, which have been published (see, e.g., Ref. 1), appear overly hazy.] It has been hypothesized[2] that the vanishing of the vacuum energy is arranged in each order of perturbation theory by corresponding Riemannian identities for theta-functions. Our purpose in the present letter is to refine this assertion and to prove it in the two-loop approximation.

1. The entity which basically determines the p-loop partition functions is a sum over all (even and odd) θ-characteristics e (Refs. 3–5):

$$\Phi = \sum_e \epsilon_e d\mu_e^{ss} \langle S_1 \ldots S_{2p-2} \rangle, \qquad (1)$$

where $d\mu_e^{ss} = \Lambda_0^{-5}\Lambda_2\Lambda_{1/2}^5[e]\Lambda_{3/2}^{-1}[e] = \det^{-5}\bar\partial_0 \det_e^5\bar\partial_{1/2}\det\bar\partial_2\det_e^{-1}\bar\partial_{3/2}$, $S_\alpha = \int_{S_p}\chi_\alpha(\psi\partial x + s_{\rm ghost})$; $\alpha = 1 \ldots 2p-2$; and χ_α are the zero modes of the gravitino field.

The p-loop partition function of a heterotic string is found by multiplying $\bar\Phi$ by the Mumford measure $d\mu_{\rm bos}$, by the modular form of weight 16 [a p-loop theta-function of the lattice $\Gamma_8\times\Gamma_8$ or Γ_{16} (Ref. 3, for example)], and by the nonholomorphic factor $(\det N_1)^{-5}$ and by integrating over the moduli space M_p. Vertex operators are introduced in the expressions for the amplitudes, in particular, in the correlation function in (1). A complete proof of supersymmetry and of the finiteness of the theory requires examining sums of more-complicated correlation functions of this sort. We will not discuss amplitudes here.

Returning to (1), we note that the sign factors $\epsilon_e = \pm 1$ can be determined if some odd θ-characteristic e_* is given. We will then have $\epsilon_e = \langle e_*, e\rangle$. The module-invariant scalar product of θ-characteristics $e_1 = [{}^{\delta_1}_{\epsilon_1}]$ and $e_2 = [{}^{\delta_2}_{\epsilon_2}]$ is determined in accordance with the rule $\langle e_1, e_2\rangle = (-)^{\vec\delta_1\vec\epsilon_2 - \vec\delta_2\vec\epsilon_1}$. The zero modes of the gravitino, χ_α, are the duals of a set of holomorphic 3/2-differentials ζ_β: $\int_{S_p}\chi_\alpha\zeta_\beta = \delta_{\alpha\beta}$. The same set of 3/2-differentials appears in the definition[6]

$$\det\bar\partial_{3/2} = \frac{\int D\eta D\zeta \prod_{\alpha=1}^{2p-2}\zeta(Q_\alpha)\exp\int\eta\bar\partial\zeta}{\det_{(\alpha\beta)}\zeta_\alpha(Q_\beta)}.$$

The determinant which has appeared in the denominator cancels out the dependence on the normalization of χ_α in (1). The zero modes χ_α have a further arbitrariness which stems from the gauge transformations $\chi_\alpha \to \chi_\alpha + \bar{\partial}\epsilon_\alpha$, which leave the χ_α the duals of the set $\{\zeta_\beta\}$. In an arbitrary gauge χ_α, the expression for Φ apparently does not vanish *before* an integration over the module space. Gauge transformations change the integrand by a total derivative with respect to the module space.[1] Using the gauge arbitrariness, we choose χ_α in the form $\delta^{(2)}(z - Q_\alpha)(d\bar{z}/dz^{1/2})$ for convenience, where the Q_α are $2p$–2 points on the Riemannian surface S_p of type p. With this choice of χ_α, expression (1) seems completely indistinguishable from the representation of Ref. 7, where the correlation function $\langle S_1 ... S_{2p-2}\rangle$ in (1) is replaced by $\langle \xi(Q_0) Q\xi(Q_1) ... Q\xi(Q_{2p-2})\rangle$ (ξ is a scalar field, and Q is a BRST operator). We show below that a correct choice of χ_α will cause the sum Φ to vanish by itself (at least in the case $p = 2$).

2. Using the odd and nonsingular θ-characteristic e_* which we introduced above, we can simplify the expression for $\Lambda_j = \det^{1/2}\bar{\partial}_0 \det \bar{\partial}_j$ dramatically and determine a convenient basis χ_α. The characteristic e_* determines the $p - 1$ points $R_1 ... R_{p-1}$ on Riemannian surface S_p: double zeros of a holomorphic 1-differential $v_*^2 = \theta_{*,i}\omega_i$. [In general, according to the Riemannian theorem, we would have $\theta_* (\mathbf{z}_1 + ... + \mathbf{z}_{p-1} - \mathbf{R}_1 - ... - \mathbf{R}_{p-1}) \equiv 0$ for all $z_1 ... z_{p-1}$ on S_p, where $\mathbf{z} = \int^z \omega$ is a Jacobi mapping, and $\omega_1 ... \omega_p$ are canonical holomorphic 1-differentials on S_p.]

As was shown in Refs. 6 and 8, if we choose the Arakelov metric $|v_*|^4$ on S_p, we find

$$\Lambda_{3/2}[e] = \det{}^{1/2}\bar{\partial}_0 \det_e \bar{\partial}_{3/2} \sim \theta_e(\Sigma \mathbf{Q}_\alpha - 2\Sigma \mathbf{R}_i) / \det \zeta_\alpha(Q_\beta). \tag{2}$$

We also know the result for $\det_e \bar{\partial}_{1/2}$ [see Eq. (4)].

We first consider the contribution of the matter fields ψ and x to the spin-current correlation function $\langle S_1 ... S_{2p-2}\rangle$:

$$\langle \psi\partial x(Q_1) \ldots \psi\partial x(Q_{2p-2})\rangle_e / \det_e[\zeta_\alpha(Q_\beta)]. \tag{3}$$

The determinant in the denominator is related to the normalization of the zero modes χ_α:

$$\delta^{(2)}(z - Q_\alpha)\frac{d\bar{z}}{d\bar{z}^{1/2}} = \sum_\beta \zeta_\alpha(Q_\beta)\chi_\beta(z).$$

Each field ψ in (3) must be treated as the sum of two fields, $\psi = \tilde{\psi} + \tilde{\tilde{\psi}}$. Since the action of fermions is of the form $\int \tilde{\tilde{\psi}}\bar{\partial}\tilde{\psi}$, the only nonzero correlation functions are those with equal numbers of fields $\tilde{\psi}$ and $\tilde{\tilde{\psi}}$. Here we have

$$\det{}^{1/2}\bar{\partial}_0 \det{}_e \bar{\partial}_{1/2} \langle \tilde{\psi}(x_1) \ldots \tilde{\psi}(x_n) \tilde{\tilde{\psi}}(y_1) \ldots \tilde{\tilde{\psi}}(y_n) \rangle_e$$

$$= \theta_e (\mathbf{x}_1 + \ldots \mathbf{x}_n - \mathbf{y}_1 - \ldots - \mathbf{y}_n) \frac{\prod_{i<j}^{n} E(x_i, x_j) E(y_i, y_j)}{\prod_{i,j} E(x_i, y_j)}$$

$$= \theta_e (0) \det{}_{(ij)} \frac{\theta_e (\mathbf{x}_i - \mathbf{y}_j) / \theta_e (0)}{E(x_i, y_j)} , \tag{4}$$

and E is the Prime bidifferential[9] $E(x,y) = \theta_* (\mathbf{x} - \mathbf{y})/\nu_* (x)\nu_* (y)$. This expression holds for any (not necessarily even) nonsingular characteristic e.

The "Lorentzian" signs μ on all of the fields ψ^μ in (4) must of course be the same ($\mu = 1...5$, since the complex dimensionality of the space-time is $10/2 = 5$). The correlation function for the fields with different values of μ splits up into a product of correlation functions. The same is true of the correlation functions of the fields ∂x^μ. Accordingly, correlation function (3) should be written as a sum over Lorentzian indices. In each term, the $2p-2$ operators $\psi(Q_\alpha)$ are divided into five groups on the basis of the values of μ:

$$\Lambda^5_{1/2} [e] \langle \psi \partial x (Q_1) \ldots \psi \partial x (Q_{2p-2}) \rangle$$

$$\to \Sigma_{\rm div} c_{\rm div} \prod_{\mu=1}^{5} \Lambda_{1/2} [e] \langle \psi^\mu (Q_{\mu 1}) \cdots \psi^\mu (Q_{\mu n_\mu}) \rangle_e$$

$$\langle \partial x^\mu (Q_{\mu 1}) \ldots \partial x^\mu (Q_{\mu n_\mu}) \rangle . \tag{5}$$

Here all the numbers n_μ are even; $\Sigma^5_{\mu=1} n_\mu = 2p-2$; and the factors $c_{\rm div}$ are combinational factors which depend on the dimensionality of the space-time, $d/2 = 5$. Each fermion correlation function in (5) contains $2^{n_\mu - 1}$ other terms, which are identifiable by the splitting of the operators ψ into $\tilde{\psi}$ and $\tilde{\tilde{\psi}}$. The final expression for Φ reduces to the following sum over the partitions of the points Q_α into pairs $(\tilde{Q}_i, \tilde{\tilde{Q}}_i)$, $i = 1...p-1$:

$$\Phi = \Lambda_2 \Lambda_0^{-5} \Sigma_{\rm div} F[\tilde{Q}_i, \tilde{\tilde{Q}}_i] \sum_e \langle e_*, e \rangle \theta_e^{6-p} (0) \frac{\prod_{i=1}^{p-1} \theta_e (\tilde{\mathbf{Q}}_i - \tilde{\tilde{\mathbf{Q}}}_i)}{\theta_e (\Sigma \mathbf{Q}_\alpha - 2 \Sigma \mathbf{R}_i)} . \tag{6}$$

The functions $F[\tilde{Q}_i, \tilde{\tilde{Q}}_i]$ *do not* depend on the θ-characteristics. They include combinations of Prime bidifferentials in addition to combinatorial factors, and they also contain contributions (not analytic in terms of moduli) which contain the imaginary part of a matrix of periods. These contributions come from the correlation functions of the fields ∂x.

3. In the case of type 2, there is only a single point R, and there is only a single pair $(Q, \tilde{Q}) = (Q_1, Q_2)$. The sum over characteristics in (6) reduces to

$$A_{1|2} = \sum_e \langle e_*, e \rangle \theta_e^4 \frac{\theta_e (\mathbf{Q}_1 - \mathbf{Q}_2)}{\theta_e (\mathbf{Q}_1 + \mathbf{Q}_2 - 2\mathbf{R})} . \tag{7}$$

Only the even nonsingular characteristics e contribute here (this point remains in force for all $p<6$). If we choose $Q_2=R$, we find that sum (7) converts into a Riemannian identity in the form[2] $\Sigma_e \langle e_*,e\rangle \theta_e^4 \equiv 0$.

This simple approach fails for types $p>2$. At $p=3$, for example, there are three different assignments of the four points Q_α to pairs and three different sums over characteristics:

$$A_{12|34}=\sum_e \langle e_*,e\rangle \theta_e^3 \theta_e(\mathbf{Q}_1-\mathbf{Q}_2)\theta_e(\mathbf{Q}_3-\mathbf{Q}_4)/\theta_e(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3+\mathbf{Q}_4-2\mathbf{R}_1-2\mathbf{R}_2);$$

$$A_{13|24}=\sum_e \langle e_*,e\rangle \theta_e^3 \theta_e(\mathbf{Q}_1-\mathbf{Q}_3)\theta_e(\mathbf{Q}_2-\mathbf{Q}_4)/\theta_e(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3+\mathbf{Q}_4-2\mathbf{R}_1-2\mathbf{R}_2);$$

$$A_{14|23}=\sum_e \langle e_*,e\rangle \theta_e^3 \theta_e(\mathbf{Q}_1-\mathbf{Q}_4)\theta_e(\mathbf{Q}_2-\mathbf{Q}_3)/\theta_e(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3+\mathbf{Q}_4-2\mathbf{R}_1-2\mathbf{R}_2). \tag{8}$$

If we set $Q_3=R_1$ and $Q_4=R_2$, we find that the dependence on Q_1 and Q_2 drops out immediately from only one linear combination of these expressions:

$$A_{13|24}/E_{13}E_{24}-A_{14|23}/E_{14}E_{23}=(E_{12}E_{34}/E_{13}E_{14}E_{23}E_{24})\sum_e \langle e_*,e\rangle \theta_e^4\equiv 0.$$

All the sums A themselves convert into Riemannian identities if we make all four of the points Q_α (instead of just two of them) the same as R_1 and R_2, e.g., $Q_1=Q_3=R_1$, $Q_2=Q_4=R_2$ [in this case we have $A_{12/34}=\Sigma_e\langle e_*,e\rangle\theta_e^2\theta_e^2(\mathbf{R}_{12})\equiv 0$; see Eq. (10)]. Setting things equal in this fashion requires some caution, however, since the functions $F[\widetilde{Q}_i,\widetilde{\widetilde{Q}}_i]$ in (6) have poles when any two of the points Q_α coincide. It can be seen from (3) that these poles are of no higher than third order in each of the variables $\vec{\xi}_1=\mathbf{Q}_1-\mathbf{Q}_3=\mathbf{Q}_1-\mathbf{R}_1;\vec{\xi}_2=\mathbf{Q}_2-\mathbf{Q}_4=\mathbf{Q}_2-\mathbf{R}_2$. The vanishing of the partition function will thus be arranged if each sum A has a zero of higher order in ξ_1 and ξ_2. If we assume that the quantities ξ_1 and ξ_2 are small quantities of the same order ($\xi_1,\xi_2\sim\xi$), then this condition will indeed hold (for $p=3$). To prove this point, we need to use the Riemannian identity $\Sigma_e\langle e_*,e\rangle\theta_e^3\theta_e(\mathbf{z})=2^p\theta_*^4(\mathbf{z}/2)$ (which can be proved by, for example, the method described in Ref. 2), from which it follows that we have

$$\sum_e \langle e_*,e\rangle\theta_e^3\theta_e\Big(\int_R^{R+\xi}\vec{\omega}\Big)\sim\theta_*^4\Big(\frac{1}{2}\int_R^{R+\xi}\vec{\omega}\Big)\sim\xi^{12}. \tag{9}$$

In the last transition we need to note that we have $\theta_{*,i}\omega_i'(R)=\theta_{*,i}\omega_i'(R)=0$ and thus $\theta_*(\frac{1}{2}\int_R^{R+\xi}\vec{\omega})\sim\xi^3$. [By virtue of the Riemannian theorem, we have $\theta_*(\int_R^{R+\xi}\vec{\omega})\equiv 0$, but we need a θ-function of half the argument, since we have only a smallness on the order of ξ instead of an identical zero. Unfortunately, the order of this smallness does not depend on p, so that this argument by itself is insufficient in the case $p>3$.] Yet another identity is necessary in the case

$$A_{12|34}:\ \sum_e\langle e_*,e\rangle\theta_e^2\theta_e(\mathbf{R}_{12})\vec{\theta}_e(\mathbf{R}_{12}+\vec{\xi})=2^p\theta_*^2(\vec{\xi}/2)\theta_*^2(\mathbf{R}_{12}+\vec{\xi}/2)\sim\xi^{10}. \tag{10}$$

Unfortunately, even at $p=3$ the situation is more complicated for an arbitrary relation between ξ_1 and ξ_2. With increasing type number, the difficulties worsen.

4. The correlation function of the ghost spin currents in (1) is an analytic function on the moduli space, but it is significantly more complicated than the fermion correlation functions. It is expressed in terms of θ-functions of various combinations of Q_α and their logarithmic derivatives. A specific expression in the case of type 2 is given in Ref. 1, among other places.[1)] Even at $p=2$ the ghost correlation function evidently does not vanish if we simply set $Q_2 = R$ and leave Q_1 arbitrary. The corresponding contribution vanishes only if we let $Q_1 \to R$. The residues at all the poles $\xi^{-3}, \xi^{-2}, \xi^{-1}, \xi^{0}$ vanish by virtue of relation (9) in this case (now we are talking about $p=2$!). It follows from relations of the type in (10) that we also find a zero result if we let Q_1 go to not R but any other of the six branch points on the hyperelliptic surface of type 2. (This point was verified in Ref. 10 by an explicit and extremely laborious calculation in hyperelliptic coordinates. See also Ref. 11 regarding calculations in these coordinates.)

5. It is not completely clear to what extent the arguments above can be generalized to higher types. The sums similar to A deserve special attention [as can be seen from Eq. 4), they enter sum (6) in combinations of such a nature that there is no singularity at $\theta_e(0) = 0$ (for $p > 6$)]. We have attempted to identify the reasons for the dramatic simplification of the expression for the partition function in the case $p=2$. We believe that in order to achieve the vanishing of the partition functions *before* the integration over moduli space, we must in any case choose χ_α to be localized at the points R_i (in the hyperelliptic case, we could evidently also use other branch points[10,11]).

We wish to stress that representation (1) of the partition function in terms of the space of even (and not super-) moduli obviously violates modular invariance through the introduction of noninvariant zero modes χ_α. Our final expressions depend on the particular choice of odd θ-characteristic e_* *and* are invariant under modular transformations which do not change e_*. This situation has been achieved through the use of the same odd characteristic in three *different* places: in the definitions of the signs of ϵ_e, the determinants det $\bar{\partial}$, and the $(-1/2, 1)$-differentials χ_α. The result is a significant simplification of the equations; a derivative with respect to the coordinates in moduli space is replaced by an identical zero. (This result can apparently be interpreted as the elimination of certain anomalies.)

A point which deserves attention is that the correlation function of the fields ∂x in (3) is not holomorphic on the moduli space: It contains the imaginary part of a matrix of periods. The actual meaning of this circumstance will apparently become clear when we reach a better understanding of the meaning of derivatives with respect to coordinates in the moduli space and of how the various representations for the partition functions and amplitudes differ from each other. Because of the nonholomorphic behavior, it is necessary to reexamine the suggestion[3] that data on modular forms be used as a basis for the vanishing of partition functions. (One attempt to implement this suggestion was undertaken in Ref. 12.)

We should mention that this study grew out of attempts to reach an understanding of the meaning of the identities which were recently found[10,13] and which lead to the vanishing of certain contributions to the hyperelliptic partition functions in superstring theory. They were originally written in terms of branch points, rather than θ-functions.

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[1)]It is proportional to $[\theta_e^5\theta_e(2\mathbf{Q}_2-2\mathbf{R})/\theta_e^2(\mathbf{Q}_1+\mathbf{Q}_2-2\mathbf{R})][\partial_i \ln\theta_e^2(2\mathbf{Q}_2-2\mathbf{R})\omega_i(Q_1)+e$—independent terms].

[1]E. Verlinde and H. Verlinde, Preprint LPTENS-39, 1986.
[2]A. Morozov and A. Perelomov, Pis'ma Zh. Eksp. Teor. Fiz. **44**, 157 (1986) [JETP Lett. **44**, 201 (1986)]; Phys. Lett. **183B**, 296 (1987).
[3]A. Belavip and V. Knizhnik, Zh. Eksp. Teor. Fiz. **91**, 364 (1986) [Sov. Phys. JETP **64**, 214 (1986)].
[4]V. Knizhnik, Phys. Lett. **178B**, 21 (1986).
[5]M. Bonini and R. Iengo, Preprint TRIEST-11 EP, 1987.
[6]V. Knizhnik, Phys. Lett. **178B**, 21 (1986).
[7]E. Martinec, Nucl. Phys. **B281**, 157 (1987).
[8]L. Alvarez-Gaume, J.-B. Bost, G. Moore, P. Nelson, and C. Vafa, Phys. Lett. **178B**, 41 (1986).
[9]J. D. Fay, Theta Function on Riemann Surfaces, Springer, 1973; D. Mumford, Tata Lectures on Theta Function, Birkhauser, 1983.
[10]V. Knizhnik, Pis'ma Zh. Eksp. Teor. Fiz. **46**, 8 (1987) [JETP Lett. **46**, 7 (1987)].
[11]D. Montano, Preprint SLAC-4319, 1987.
[12]G. Moore, J. Harris, P. Nelson, and I. Singer, Phys. Lett. **178B**, 167 (1986).
[13]A. Morozov, Preprint ITEP-69, Institute of Theoretical and Experimental Physics, 1987.

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