Higher-order integrals of motion in two-dimensional models of the field theory with a broken conformal symmetry

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The "minimal" models of the two-dimensional conformal field theory, which are perturbed by the field $\Phi_{(1,3)}$ or $\Phi_{(1,2)}$, are found to have higher-order integrals of motion.

The critical points of statistical systems are known to be described by fixed renormalization-group points,[1] i.e., by conformally invariant solutions of the field theory. Several exact solutions of the conformal field theory in a two-dimensional space have recently been found (see the review by Zamolodchikov[2]). Exact correlation functions

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can be used to study the neighborhood of a fixed point on the basis of perturbation theory. The perturbation can be represented in the form $\Sigma_a \lambda^a \int \Phi_a(x) d^2x$, where Φ_a are local spin-zero fields (we will consider here only the isotropic theories) which are found in the operator algebra[2,3] A of the unperturbed conformal field theory, and λ^a are the dimensional constants (the "coupling constants"). If the field Φ_a has a right-hand and left-hand dimensionality[2,3] (Δ_a, Δ_a), then the scalar constant will satisfy the condition $\lambda^a \sim R^{\Delta_a - 1}$ (R is the length). We can say that λ^a has the dimensionalities $(1-\Delta_a, 1-\Delta_a)$, so that the perturbation is dimensionless. The conformal invariance of the unperturbed theory is equivalent to the existence at $\lambda^a = 0$ of an infinite set of the integrals of motion $L_n, \bar{L}_n$, $n = 0, \pm 1, \pm 2,...$—conformal-transformation generators which generate the Virasoro[3] algebra and which can be expressed in terms of the components $T = T^{zz}$ and $\bar{T} = T^{\bar{z}\bar{z}}$ of a traceless (at $\lambda = 0$) stress tensor

$$2\pi i L_n = \oint dz z^{n+1} T(z); \quad 2\pi i \bar{L}_n = \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}), \tag{1}$$

where $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$, and (x^1, x^2) are the Cartesian coordinates $\mathbf{R}^2$. If $\lambda \neq 0$, the conformal invariance is generally violated and operators (1) are no longer integrals of motion. The components of the momentum, $P = L_{-1}$ and $\bar{P} = \bar{L}_{-1}$, and of the angular momentum, $M = L_0 - \bar{L}_0$, are the exception. The conservation of these components is assured by the Euclidean invariance of the theory. In this letter we will show, however, that in several cases the perturbation theory has "higher" commutative integrals of motion and is likely fully integrable.

The operator algebra A of any conformal field theory contains the conformal class[3] of a unit operator $[I] = \mathbf{T} \times \bar{\mathbf{T}}$, where all the space vectors $\mathbf{T}(\bar{\mathbf{T}})$ are found by applying the operators $L_n (\bar{L}_n)$ with $n < 0$ to I; specifically, $T = L_{-2} I \in \mathbf{T}$. Clearly, $\mathbf{T} = \oplus_{s=0}^{\infty} \mathbf{T}_s$, where any field $T_s \in \mathbf{T}_s$ has the dimensionlities $(s,0)$ (i.e., spin s). For example, $\mathbf{T}_1 = 0$ (since $L_{-1} I = 0$), $\mathbf{T}_0$, $\mathbf{T}_2$, and $\mathbf{T}_3$ are one-dimensional (in which I, T, and $\partial_z T$, respectively, are formed), $\mathbf{T}_4$ contains the field $T_4 = [L_{-2}^2 - (3/5) L_{-4}] I$, in addition to $\partial_z^2 T$; $\mathbf{T}_5$ is stretched around $\partial_z^3 T$ and $\partial_z T_4$, and $\mathbf{T}_6$ is stretched around $\partial_z^4 T$, $\partial_z^2 T_4$, and

$$T_6^{(1)} = \left(L_{-2}^3 - \frac{1}{3} L_{-3}^2 - \frac{19}{15} L_{-4} L_{-2} - \frac{2}{3} L_{-6}\right) I;$$

$$T_6^{(2)} = \frac{1}{9} \left(-\frac{5}{2} L_{-3}^2 + 4 L_{-4} L_{-2} + \frac{10}{7} L_{-6}\right) I.$$

The field T_4 appears in the regular part of the operator expansion of $T(z)T(0)$ and may be regarded as a (regularized) square of T: $T_4 \sim :T^2:$. In the same sense, $T_6^{(1)} \sim :T^3:$, $T_6^{(2)} \sim :\partial_z T \partial_z T:$. Each field $T_s \in \mathbf{T}_s$ (in particular, T_4 and T_6) is a "conserved current": $\partial_{\bar{z}} T_z = 0$, i.e., $T_s = T_s(z)$. It generates a set of (dependent) integrals of motion which can be polynomially expressed in terms of L_n. We show below that in certain cases some of these integrals of motion "survive" in the perturbation theory.

A series M_p, $p = 3,4,5,...$ of "minimum" unitary solutions of the conformal field theory, which correspond to the values $c_p = 1 - [6/p(p+1)]$ of the central charge in the Virasoro algebra, was constructed in Refs. 3 and 4. The operator algebra M_p

contains $p(p-1)/2$ conformal classes $[\Phi_{(n,m)}]$, $n=1,2,...,p-1$, $m=1,2,...,p$; $\Phi_{(p-n,p+1-m)}=\Phi_{(n,m)}, \Phi_{(1,1)}=I$. The dimensionalities of the fields $\Phi_{(n,m)}$ are given by the Kac formula.[3] The operator algebra M_p contains the subalgebra $A_{(1,*)} = \oplus_m [\Phi_{(1,m)}]$, which in turn contains the subalgebra $A_{(1,**)} = \oplus_e [\Phi_{(1,2l+1)}]$. We will examine two types of perturbation theory:

$$H_\lambda = H^{(p)} + \lambda \int \Phi(x) d^2 x; \tag{2a}$$

$$H_\mu = H^{(p)} + \mu \int \phi(x) d^2 x, \tag{2b}$$

where $H^{(p)}$ in each case is the action of the conformal field theory of M_p, $\Phi \equiv \Phi_{(1,3)}$, and $\phi \equiv \Phi_{(1,2)}$. The field $\Phi_{(1,3)}$ ($\Phi_{(1,2)}$) is distinguished by the fact that it has the smallest dimensionality (if the unit operator is eliminated) in $A_{(1,**)}$ in $A_{(1,*)}$. The fields Φ and ϕ have the dimensionalities (Δ,Δ) and (δ,δ), respectively, where $\Delta = 1-\epsilon$, and $\delta = (1/4 - 3\epsilon/8)$; $\epsilon = 2/(p+1)$. Since $\Delta < 1$ and $\delta < 1$ for all values of p, the perturbation theories for (2a) and (2b) do not have ultraviolet divergences, and the ultraviolet asymptotic solution of each theory is described by the conformal field theory of M_p. The fields Φ and ϕ are degenerate fields which satisfy the equations[3]

$$\left(L_{-3} - \frac{2}{\Delta+1} L_{-1} L_{-2} + \frac{1}{(\Delta+1)(\Delta+2)} L_{-1}^3\right)\Phi = 0; \tag{3a}$$

$$\left(L_{-2} - \frac{3}{2(2\delta+1)} L_{-1}^2\right)\phi = 0. \tag{3b}$$

The conservation of momentum in (2) is assured by the equations $\partial_\mu T^{\mu\nu} = 0$, i.e.,

$$\partial_{\bar z} T = \delta_z \Theta; \ \partial_z \bar T = \partial_{\bar z} \Theta, \tag{4}$$

where $\Theta = -T^\mu_\mu$ is the trace of the stress tensor; here $\Theta = \lambda\epsilon\Phi$ in (2a) and $\Theta = \mu(1-\delta)\Phi$ in (2b). We will show that (2a) and (2b) have "higher" integrals of motion.

Let us first examine (2a). At $\lambda \neq 0$ the derivative $\partial_{\bar z} T_4$ can be written in the form $\Sigma_{k=1}^{\infty} \lambda^k Q_2^{(k)}$, where $Q_2^{(k)}$ are local fields. As can be seen from the structure of the perturbation-theory series, $Q_2^{(k)} \in A_{(1,**)}$. The fields $Q_2^{(k)}$ must have spin 3 and dimensionalities $(4-k\epsilon, 1-k\epsilon)$. For example, $Q_2^{(1)}$ can be only a linear combination of the fields $L_{-3}\Phi$, $L_{-1}L_{-2}\Phi$, and $L_{-1}^3\Phi$. By virtue of (3a), however, these fields are linearly dependent and we have $Q_2^{(1)} = \partial_z(a_1 L_{-2} + a_2 L_{-1}^2)\Phi$ (since $L_{-1} = \delta_z$), where a_1 and a_2 are constants whose values are now not important. In addition, $A_{(1,**)}$ generally does not contain fields with the dimensionalities $(4-k\epsilon, 1-k\epsilon)$ with $k > 1$, with the exception of the case $k = (p+1)/2$, which is possible only when p is odd. In this case we have $Q_2^{[(p+1)/2]} \sim \partial_z T$. Consequently, the following exact equation in perturbation theory (2a) is valid:

$$\partial_{\bar z} T_4 = \partial_z \Theta_2, \tag{5}$$

where $\Theta_2 = \lambda(a_1 L_{-2} + a_2 L_{-1}^2)\Phi + \lambda^{[(p+1)/2]}\alpha T$ ($\alpha = 0$ if $p \in 2\mathbf{T}$). Each one of the derivatives $\partial_{\bar z} T_6^{(i)}$, $i = 1,2$, has the form $\Sigma_{k=1}^{\infty} \lambda^k Q_4^{(i)(k)}$, where $Q_4^{(i)(k)} \in A_{(1,**)}$ has the dimensionalities $(6 - k\epsilon, 1 - k\epsilon)$. The fields $Q_4^{(i)(1)}$, for example, may, in addition to the total derivatives with respect to z (i.e., the terms that contain L_{-1}), have only the terms $L_{-5}\Phi$ and $L_{-2}L_{-3}\Phi$. Here the second term can be eliminated by means of (3a). There is therefore a linear combination $T_6 = T_6^{(1)} + LT_6^{(2)}$ such that

$$\partial_{\bar z} T_6 = \partial_z \Theta_4, \tag{6}$$

where $\Theta_4 = \lambda(b_1 L_{-4} + b_2 L_{-1}^2 L_{-2} + b_3 L_{-1}^4)\Phi + \lambda^{[(p+1)/2]}(\beta_1 T_4 + \beta_2 \partial_z^2 T)$, and b and β are constants. A more exact calculation gives $h = (28 + 5c_p)/30$, where c_p is the value of the central charge in M_p. It follows from (4)–(6) that perturbation theory (2a) has integrals of motion:

$$P_{2l+1} = \oint (T_{2l+2} dz + \Theta_{2l} d\bar z), \tag{7}$$

where $l = 0,1,2$; $P_1 \equiv P$ is the "right-hand" component of the momentum, and integrals of motion in (7) with $l = 1,2$ are higher-order integrals of motion. There are of course similar "left-hand" integrals of motion. Direct calculations show that all the operators in (7) are commutative operators. It can be assumed that in (2a) there is an infinite series of integrals of motion such as those in (7) with $l = 1,2,3,\ldots$, although these operators with $l > 2$ have not yet been constructed. Zamolodchikov[5] showed that model (2a) with $\lambda > 0$ and a large enough p has a conformally invariant infrared asymptotic solution which is described by the conformal field theory of M_{p-1}. At $\lambda < 0$, there is no particular reason to expect the appearance of a zero in the β function, so that such a theory apparently has a finite correlation radius, $R_c \sim \lambda^{-1/e}$, and massive particles. If this is true, integrals of motion in (7) account in this theory for the elastic nature of particle scattering, for the factorization of the S matrix, and for other features.[6]

Let us now consider (2b). Here $\partial_{\bar z} T_4$ cannot be a total derivative with respect to z because of the presence of the term $L_{-3}\phi$ which appears in the first order in μ. In first order in μ the derivatives $\delta_{\bar z} T_6^{(1)}$ and $\delta_{\bar z} T_6^{(2)}$ may contain "dangerous" terms (i.e., terms which cannot be reduced to the derivatives with respect to z) such as $L_{-5}\phi$ and $L_{-3}L_{-2}\phi$. The second term, however, can be dropped because of (3b). In (2b), therefore, there is a field $T_6' = T_6^{(1)} + h' T_6^{(2)}$, which satisfies (6) with $T_6 \to T_6'$ and $\Theta_4 \to \Theta_4' = \mu(b_1' L_{-4} + b_2' L_{-1} L_{-3} + b_3' L_{-1}^4)\phi + \mu^{[4(p+1)/3]}(\beta_1' T_4 + \beta_2' \partial_z^2 T)$, where the coefficients β' may be nonvanishing only if $p + 1 = 0$ (model 3). Perturbation theory (2b) thus has a "higher-order" integral of motion (7) with $l = 2$. We assume that model (2b) has an infinite series of commuting integrals of motion. In any case, integral of motion (7) with $l = 2$ alone can account for the purely elastic nature of the particle scattering in (2b).

The validity of the argument in the preceding paragraph will hold if we set $\phi = \Phi_{(2,1)}$ in (2b), so that such a perturbation theory also has a higher-order integral of motion for all values of p. In the special case $p = 5$, this model describes a three-position Potts model in the scaling limit[8] $T \to T_c$.

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