Hamiltonian structure of the antisymmetric action of a string

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(Submitted 29 June 1987)
Pis'ma Zh. Eksp. Teor. Fiz. **46**, No. 6, 208–210 (25 September 1987)

A Hamiltonian formulation is offered for the dynamics of a string which is described by an antisymmetric action.

An antisymmetric representation of the action of strings and superstrings[1] has recently become the subject of active research.[2-4] However, a systematic Hamiltonian formalism, which would be required for quantization, has not been constructed because of the complex structure of the couplings. One attempt[4] to move in this direction contains errors. In the present letter we analyze the structure of the couplings. We transform from a Lagrangian formulation of an antisymmetric action to a Hamiltonian formulation. The couplings found form an open algebra in terms of the Poisson brackets.

The action[1] in a D-dimensional space $\mathbf{x}$ can be written

$$S = \gamma \iint d\tau d\sigma \left\{ \frac{1}{2} (n_i^A \epsilon^{ik} n_k^B) \partial_\mu x_A \epsilon^{\mu\nu} \partial_\nu x_B - \frac{1}{2} E^{ik} (\mathbf{n}_i \mathbf{n}_k - \eta_{ik}) \right\} . \quad (1)$$

The couplings which follow from the definition of the canonical momenta for the fields $\mathbf{x}$, $\mathbf{n}_i$, and E_{ik} are

$$\vec{\Phi} = \vec{\mathcal{P}} - \gamma \mathbf{n}_i \epsilon^{ik}(\mathbf{n}_k \mathbf{x}') \approx 0, \quad \vec{\pi}_i \approx 0, \qquad \pi_{ik} \approx 0.$$

The Hamiltonian is equal to the sum of the couplings:

$$H = \int d\sigma \left\{ \frac{\gamma}{2} E^{ik} (\mathbf{n}_i \mathbf{n}_k - \eta_{ik}) - \mathbf{u}\vec{\Phi} - \mathbf{v}^i \vec{\pi}_i - v^{ik}\pi_{ik} \right\} . \tag{2}$$

Analysis of the conditions under which couplings (2) are consistent leads to four couplings of the first kind,

$$Q_0 = (\mathbf{n}_i \mathbf{n}_i - 2)\gamma - 2(\mathbf{n}_i \vec{\Phi})\epsilon^{ik}(\mathbf{n}_k \mathbf{x}')\frac{1}{\mathbf{x}'^2} + 2\frac{1}{x'^2}\vec{\pi}_i \epsilon^{ik} \mathbf{n}'_{k\perp} \approx 0,$$
$$T_0 = (\vec{\mathcal{P}} \cdot \mathbf{x}') + (\vec{\pi}^i \mathbf{n}'_{i\perp}) \approx 0, \quad \epsilon^{ik}(\vec{\pi}_i \cdot \mathbf{n}_k) \approx 0, \quad \pi_{00} - \pi_{11} \approx 0, \tag{3}$$

and $4D+2$ couplings of the second kind,

$$\mathbf{x}'_\perp = \mathbf{x}' - \mathbf{n}_i(\mathbf{n}^i \mathbf{x}') \approx 0, \qquad \vec{\mathcal{P}}_\perp = \vec{\mathcal{P}} - \mathbf{n}_i(\mathbf{n}_i \cdot \vec{\mathcal{P}}) \approx 0, \tag{4}$$

$$(\mathbf{n}_i \mathbf{n}_k) - \eta_{ik} \approx 0, \quad \vec{\pi}_i \approx 0 \tag{5}$$

$$\pi_{00} + \pi_{11} \approx 0, \qquad E_{00} + E_{11} \approx 0, \qquad E_{01} \approx 0, \qquad \pi_{01} \approx 0.$$

Couplings (4) can be solved explicitly, and the variables $\mathbf{n}_0$, $\mathbf{n}_1$ can be expressed in terms of $\mathbf{x}'$ and $\vec{\mathcal{P}}$:

$$\mathbf{n}_0 = \frac{\vec{\mathcal{P}}}{\sqrt{-\vec{\mathcal{P}}^2}}\cosh\Theta + \frac{\mathbf{x}'}{\sqrt{\mathbf{x}'^2}}\sinh\Theta ,$$

$$\mathbf{n}_1 = \frac{\mathbf{x}'}{\sqrt{\mathbf{x}'^2}}\cosh\Theta + \frac{\vec{\mathcal{P}}}{\sqrt{-\vec{\mathcal{P}}^2}}\sinh\Theta .$$

Using this circumstance along with couplings (5), we can eliminate from consideration all degrees of freedom except those described by the pairs $\mathbf{x}$, $\vec{\mathcal{P}}$ and E, π where $E = (1/2)\mathrm{Sp}E_{ik}$ is the scalar density on the world sheet, and $\pi = (1/2)\mathrm{Sp}\pi_{ik}$ is the corresponding canonical momentum. After the elimination of nonphysical degrees of freedom, Hamiltonian (2) becomes

$$H = \int d\sigma \left\{ u_0 (\mathbf{x}' \cdot \vec{\mathcal{P}}) + E\left(\sqrt{\frac{-\vec{\mathcal{P}}^2}{\mathbf{x}'^2}} - \gamma\right) + \dot{E}\pi \right\}, \tag{6}$$

where[1)] $\acute{\mathbf{x}}^2 \neq 0$. The nonvanishing simultaneous Poisson brackets of the couplings of the first kind, $T = (\mathbf{x}\cdot\vec{\mathscr{P}}) \approx 0$, $Q = (\sqrt{-\vec{\mathscr{P}}^2/\acute{\mathbf{x}}^2}) - \gamma \approx 0$, take the form

$$\{T(\sigma),\ T(\sigma')\} = [T(\sigma) + T(\sigma')]\delta'(\sigma-\sigma'),$$
$$\{Q(\sigma),\ T(\sigma')\} = Q'(\sigma)\delta(\sigma-\sigma'), \tag{7}$$
$$\{Q(\sigma),\ Q(\sigma')\} = \left[T(\sigma)\frac{1}{\acute{\mathbf{x}}^4(\sigma')} + T(\sigma')\frac{1}{\acute{\mathbf{x}}^4(\sigma')}\right]\delta'(\sigma-\sigma').$$

The coupling Q replaces the second coupling $\vec{\mathscr{P}}^2 + \gamma^2\acute{\mathbf{x}}^2 \approx 0$ in the Nambu formulation. The coupling Q is one of the roots of the coupling $\vec{\mathscr{P}}^2 + \gamma^2\acute{\mathbf{x}}^2 \approx 0$, since $(-\vec{\mathscr{P}}^2 - \gamma^2\acute{\mathbf{x}}^2) = \acute{\mathbf{x}}^2[\sqrt{(-\vec{\mathscr{P}}^2/\acute{\mathbf{x}}^2)} + \gamma]Q \approx 0$. Using $-\vec{\mathscr{P}}^2 = \gamma E^{-1}[-\dot{\mathbf{x}}\acute{\mathbf{x}}^2 + \acute{\mathbf{x}}(\dot{\mathbf{x}}\acute{\mathbf{x}})]$ and $u_0 = (\dot{\mathbf{x}}\acute{\mathbf{x}})/\acute{\mathbf{x}}^2$, we find a reparametrization-invariant action for H in (6)

$$S = \gamma \int\int d\tau d\sigma \left\{\frac{-\Sigma}{E} + E - \sqrt{-\Sigma}\right\}, \tag{8}$$

since E and $\sqrt{-\Sigma} = \sqrt{-\dot{\mathbf{x}}^2\acute{\mathbf{x}}^2 + (\dot{\mathbf{x}}\acute{\mathbf{x}})^2}$ transform as scalar densities. On the physical value of the root $E = \sqrt{-\Sigma}$ of the equations of motion for the density E, action (8) becomes the Nambu action

$$S = \gamma \int\int d\tau d\sigma \sqrt{-\dot{\mathbf{x}}^2\acute{\mathbf{x}}^2 + (\dot{\mathbf{x}}\acute{\mathbf{x}})^2}.$$

Representation (8) incorporates a Liouville mode in the form of the density E. The quantization of the couplings and action in (7) and (8) may lead to some new moments. For example, an analog of action (8) for a scalar particle

$$S_r = \int d\tau \left\{\frac{\dot{\mathbf{x}}^2}{2e} - \frac{em^2}{2} - m\sqrt{-\dot{\mathbf{x}}^2}\right\}$$

on any of the roots of the equation $-\vec{\mathscr{P}}^2 = (m + \sqrt{-\mathbf{x}^2/e^2})^2$ generates a coupling $(\sqrt{-\vec{\mathscr{P}}^2} \pm 2m) \approx 0$, which is characteristic of a Dirac particle.

A boundary condition for action (1) in the case of an open string is $\dot{\mathbf{x}}(\dot{\mathbf{x}}\acute{\mathbf{x}}) - \acute{\mathbf{x}}(\dot{\mathbf{x}}^2)(-\Sigma)^{-1/2}|_{\sigma=0,\pi} = 0$; it leads to a 0/0 indeterminate form.[2)] This difficulty can be eliminated by adding[5] to action (1) a two-dimensional Wess-Zumino term

$$\Delta S = c\int\int d\tau d\sigma \partial_\mu \mathbf{n}_i \epsilon^{\mu\nu} \partial_\nu \mathbf{n}_k .$$

The new particle-like representation found above for a Schild null string make it possible to introduce some new entities: a null membrane, a null superstring, and a null supermembrane. The action for the null membrane is

$$S_M = \gamma \int\int\int d\tau d\sigma\, d\rho \left(\frac{-\Sigma}{E}\right), \qquad \Sigma = \det(\partial_\mu \mathbf{x}\partial_\nu \mathbf{x}),$$

and in the gauge $(\dot{\mathbf{x}}\acute{\mathbf{x}}) = (\dot{\mathbf{x}}\partial_\rho \mathbf{x}) = 0$, $E = \acute{\mathbf{x}}^2(\partial_\rho \mathbf{x})^2 - (\acute{\mathbf{x}}\partial_\rho \mathbf{x})^2$ it leads to a linear equation of motion $\ddot{\mathbf{x}} = 0$, as in the case of a massless particle. The action for a null superstring or null supermembrane is found from the functionals discussed above by means of a simple replacement of all the derivatives $\partial_\mu(x)$ in Σ by superinvariant forms[1] $\vec{\Phi}_\mu = \partial_\mu \mathbf{x} - \frac{1}{4}\overline{\Theta}\vec{\gamma}\partial_\mu \Theta$. A null superstring with (or without) Wess-Zumino terms is an alternative version of the Green-Schwarz superstring to the same extent that a Schild string is an alternative version of a Nambu string. It may turn out to be noncontradictory in an arbitrary space-time dimensionality.

I wish to thank D. V. Volkov, A. I. Vaĭnshteĭn, V. D. Gershun, L. N. Lipatov, A. Yu. Morozov, and V. V. Nesterenko for interesting discussions. I also thank I. V. Tyutin for constructive comments regarding the general structure of couplings.

[1)]The case $\mathbf{n}_0^2 = 0 = \vec{\mathscr{P}}^2$ corrresponds to a Schild string[3]: $HS = \int d\sigma[u_0(\acute{\mathbf{x}}\vec{\mathscr{P}}) + E(-\vec{\mathscr{P}}^2/2\gamma\acute{\mathbf{x}}^2) + \dot{E}\pi]$, $S_S = (\gamma/2)\int\int d\epsilon d\sigma[(-\Sigma)/E]$.

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Translated by Dave Parsons