Representation of a Dirac particle as a sum over paths

A. V. Marshakov and V. Ya. Faĭnberg
P. N. Lebedev Physics Institute, Academy of Sciences of the USSR, Moscow

(Submitted 21 August 1987)
Pis'ma Zh. Eksp. Teor. Fiz. **46**, No. 7, 253–255 (10 October 1987)

A Dirac particle in an external electromagnetic field is represented as a sum over propagator paths on the basis of a local supersymmetry.

The relativistic propagator of a particle is one of the basic entities which must be known in order to construct a second-quantized theory of interacting fields. The representation of the propagator in terms of a first-quantized path integral is particularly useful in cases in which the "second-quantized" action is not known, e.g., in string theory.[1] In this letter we derive a representation for a Dirac propagator on the basis of a locally supersymmetric and reparametrization-invariant action.[2] The presence of a local supersymmetry distinguishes this representation from those offered previously.[3]

The first-quantized action which is written for particles with spin and which has local symmetries is the action of one-dimensional supergravity:

$$S_0 = \int_0^1 dt\,\frac{1}{2}\left[e^{-1}\left(\frac{dx_\mu}{dt}\right)^2 - i\psi_\mu\frac{d\psi^\mu}{dt} - ie^{-1}\chi\psi_\mu\frac{dx^\mu}{dt}\right]. \quad (1)$$

The action S_0 corresponds to a massless particle. For a particle which has a mass and a spin, we need to add mass-dependent terms:

$$S_m = S_0 - \int_0^1 dt\,\frac{1}{2}\left[em^2 + i\psi_5\frac{d\psi_5}{dt} + im\chi\psi_5\right]. \quad (2)$$

(See Ref. 2 regarding the form of the symmetry transformations.) Conserved quantities here are the components of the angular momentum tensor,

$$M_{\mu\nu} = x_\mu\frac{dx_\nu}{dt} - x_\nu\frac{dx_\mu}{dt} + \frac{i}{2}[\psi_\mu, \psi_\nu], \quad (3)$$

which can be represented in a natural way as the sum of orbital and spin parts. The particle determined by action (1) or (2) has a spin of 1/2, if the solutions of the classical equations of motion are $\psi_\mu = \psi_\mu^0 = \gamma_\mu$, or $\gamma_5\gamma_\mu$, in the case with mass ($\gamma_5 \equiv \gamma_{D+1}$). The choice of boundary conditions for the anticommuting variables, which corresponds to the Dirac γ matrices, is important for determining the path integral.

For a massless Dirac particle, the path integral is (ψ_μ and χ are Grassmann variables)

$$\int DeD\chi \int_{\psi_\mu=\psi^0_\mu+\psi^q_\mu} D\psi \int_{x(0)=x_1}^{x(1)=x_2} Dx \exp(-S_0) \sim \int_0^\infty dT \int d\chi \int D\psi Dx$$

$$\times \exp(-S_0)\Big|_{\substack{e=T\\ \chi=X}} \sim \int \frac{d^Dp}{(2\pi)^D} e^{ip(x_2-x_1)} \frac{i\psi^0_\mu p_\mu}{p^2} \equiv \hat{G}(x_2, x_1). \quad (4)$$

Expression (4) is the propagator of a massless Dirac particle when the substitution $\psi^0_\mu \to \gamma_\mu$ is made.[1)] In the case of a particle with mass, we choose, by analogy with (4), the gauge $e = T =$ const (an analog of modules in string theory[1)]) and $\chi = X =$ const (an analog of supermodules). We then write

$$\int DeD\chi \int_{\substack{\psi_\mu=\psi^0_\mu+\psi^q_\mu\\ \psi_5=\psi^0_5+\psi^q_5}} D\psi \int_{x_1}^{x_2} Dx \exp(-S_m)$$

$$\sim \int \frac{d^Dp}{(2\pi)^D} e^{ip(x_2-x_1)} \frac{i\psi^0_\mu p_\mu - m\psi^0_5}{p^2+m^2}. \quad (5)$$

After the replacements $\psi^0_\mu \to \gamma_5\gamma_\mu$ and $\psi^0_5 \to \gamma_5$, expression (5) corresponds to a Dirac propagator.

We now consider the case in which a particle is moving in an external electromagnetic field. Our problem is to derive a representation of a spinor Green's function in the external field which is analogous to the representation for a scalar particle:

$$G(x_2, x_1 | A) = \int De \int_{x_1}^{x_2} Dx \exp\left(-\int_0^1 dt\left[\frac{1}{2}e^{-1}\left(\frac{dx_\mu}{dt}\right)^2 - ig\frac{dx^\mu}{dt}A_\mu(x(t))\right]\right). \quad (6)$$

Brink *et al.*[2] have written a globally supersymmetric Lagrangian for the interaction of a Dirac particle with an electromagnetic field:

$$L_A = g\left[\frac{dx^\mu}{dt}A_\mu(x(t)) + \frac{i}{2}F_{\mu\nu}(x(t))\psi^\mu\psi^\nu\right]. \quad (7)$$

Lagrangian (7) can be generalized in a natural way to the case of a local supersymmetry. The total action is

$$S_A = S_0 + \int_0^1 dt g\left[\frac{dx^\mu}{dt}A_\mu(x(t)) + \frac{i}{2}e(t)F_{\mu\nu}(x(t))\psi^\mu\psi^\nu\right]. \quad (8)$$

By analogy with the scalar case, we can construct a propagator for a spinor particle in an external electromagnetic field for a first-order equation, specifying appropriate boundary conditions on $x_\mu(t)$ and $\psi_\mu(t)$. In Euclidean terms $[e^{-1}(t) \to ie^{-1}(t)]$ we write

$$\hat{G}(x_2, x_1 | A) = \int DeD\chi D\psi Dx \exp(-S_A). \quad (9)$$

Expressions (6) and (9) cannot be evaluated for an arbitrary external field. Let us attempt to determine some of their properties by perturbation theory. For this purpose we define the variational derivatives

$$j_\mu(x_2, x_1 | x) = -\frac{i}{g} \frac{\delta}{\delta A_\mu(x)} G(x_2, x_1 | A)|_{A_\mu = 0}, \tag{10a}$$

$$\hat{j}_\mu(x_2, x_1 | A) = -\frac{i}{g} \frac{\delta}{\delta A_\mu(x)} \hat{G}(x_2, x_1 | A)|_{A_\mu = 0}. \tag{10b}$$

The divergences of expressions (10) reduce to a calculation of free propagators:

$$\begin{aligned} \frac{\partial}{\partial x_\mu} j_\mu(x_2, x_1 | x) &= \delta(x - x_1) G(x_2, x) - \delta(x_2 - x) G(x, x_1) \\ \frac{\partial}{\partial x_\mu} \hat{j}_\mu(x_2, x_1 | x) &= \delta(x - x_1) \hat{G}(x_2, x) - \delta(x_2 - x) \hat{G}(x, x_1), \end{aligned} \tag{11}$$

where $\hat{G}(x_2,x_1)$ is defined in (4), and $G(x_2,x_1)$ is[1]

$$G(x_2, x_1) = G(x_2, x_1 | A)\Big|_{A_\mu = 0} = 2\int \frac{d^D p}{(2\pi)^{D/2}} \frac{e^{ip(x_2 - x_1)}}{p^2}. \tag{12}$$

The expressions $j_\mu(x_2,x_1|x)$ and $\hat{j}_\mu(x_2,x_1|x)$ can be evaluated directly from the path representation. They reduce to the expressions

$$j_\mu(x_2, x_1 | x) \sim G(x_2, x) \frac{\overleftrightarrow{\partial}}{\partial x_\mu} G(x, x_1), \tag{13a}$$

$$\hat{j}_\mu(x_2, x_1 | x) \sim \hat{G}(x_2, x) \gamma_\mu \hat{G}(x, x_1). \tag{13b}$$

For the derivation of (13b), we choose the gauge $\chi = X = \text{const}$ and $e = T = \text{const}$ in (9) and (10b); carrying out the integration over X, we find

$$\hat{j}_\mu(x_2, x_1 | x) \sim \langle \int_0^\infty \frac{dT}{T} \int_0^1 d\tau \dot{x}_\lambda(\tau) \psi_\lambda(\tau)$$

$$\times \int_0^1 dt \left(\dot{x}_\mu(t) + \frac{T}{2} [\psi_\mu(t) \psi_\nu(t)]_- \frac{\partial}{\partial x_\nu} \right) \delta(x - x(t)) \rangle_{\psi, x}.$$

Taking an average over ψ, we find

$$\hat{j}_\mu(x_2, x_1 \mid x) \sim \langle \int_0^\infty dT \int_0^1 d\tau \int_0^1 dt \dot{x}_\lambda(\tau) \left\{ \gamma_\lambda \left(\frac{\partial}{\partial x_\mu^1} - \frac{\partial}{\partial x_\mu^2} \right) + \epsilon(t-\tau)(\gamma_\nu \delta_{\mu\lambda} - \gamma_\mu \delta_{\nu\lambda}) \frac{\partial}{\partial x_\nu} + (\gamma_\lambda \gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu \gamma_\lambda) \frac{\partial}{\partial x_\nu} \right\} \delta(x - x(t)) \rangle_x ,$$

$$[\gamma_\mu, \gamma_\nu]_+ = -\delta_{\mu\nu}, \quad \epsilon(t-\tau) = \frac{1}{2}[\theta(t-\tau) - \theta(\tau - t)] .$$

Integrating over $x_\mu(t)$, we can find (13b).

1)This representation was derived by Dotsenko[4] on the basis of a local supersymmetry.

[1]A. Cohen, G. Moore, P. Nelson, and J. Polchinski, Nucl. Phys. **B267**, 143 (1986).
[2]L. Brink, P. Di Vecchia, P. Howe, S. Deser, and B. Zumino, Phys. Lett. **64B**, 435 (1976); L. Brink, P. Di Vecchia, and P. Howe, Nucl. Phys. **B118**, 76 (1977).
[3]W. Tobocman, Nuovo Cim. **3**, 1213 (1956); G. V. Ryazanov, Zh. Eksp. Teor. Fiz. **33**, 1439 (1957) [*sic*]; E. S. Fradkin, Nucl. Phys. **76**, 588 (1966); I. A. Batalin and E. S. Fradkin, Teor. Mat. Fiz. **5**, 190 (1970); P. Di Vecchia and F. Ravndal, Phys. Lett. **73A**, 371 (1979); F. Bordi and R. Cassalbuoni, Phys. Lett. **93B**, 308 (1980); M. Henneaux and C. Teitelboim, Ann. Phys. **143**, 127 (1982).
[4]Vl. S. Dotsenko, Nucl. Phys. **B285** [FS19], 45 (1987).

Translated by Dave Parsons