Order-parameter function in a spin glass on a Bethe lattice

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An analysis of a class of hierarchical boundary conditions for a spin glass on a Bethe lattice with an infinite branching leads to the appearance, below the transition temperature, of a continuous order-parameter function $S(x)$, which is monotonic at $x \in [0, 1]$.

A transition to a nonergodic phase below the de Almeida–Thouless temperature[2] is known to occur in the model of a spin glass with an infinite spin-spin interaction radius, as proposed by Sherrington and Kirkpatrick.[1] According to Parisi,[3,4] this transition consists, from the mathematical standpoint, of the replacement of the Edwards-Anderson parameter[5] q by a continuous function $q(x)$, which is monotonic for

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$x \in [0,1]$. This function describes the overlap in phase space of valleys separated by a distance x (Ref. 6).

Since the properties of real spin glasses differ from those of the Sherrington-Kirkpatrick model, other models have recently come under study. Interesting in this regard is the Ising spin glass on a Bethe lattice.[7,8] As was shown in Ref. 8, the transition to a spin-glass phase is characterized in this model by the appearance of a dependence of the effective fields h_i at site i of the Bethe lattice on the boundary conditions. Working on the basis of the ideas of Ref. 8, de Oliveira and Salinas[9] derived an expression in the infinite-branching limit ($r \to \infty$) for the overlap: $S = \overline{m_{i1} m_{i2}}$, where $m_i = \tanh h_i$, the superior bar means an average over the distribution of exchange integrals J_{ij}, and the subscripts 1 and 2 refer to two different boundary conditions (two replicas[8]). The solution $S = \widetilde{S} < q = \overline{m_i^2}$ turns out to be stable below a temperature given by the same expression as that for the de Almeida–Thouless temperature in the Sherrington-Kirkpatrick model. Consequently, in contrast with the Sherrington-Kirkpatrick model, only one more parameter, $\widetilde{S}$, was found in addition to q on the Bethe lattice in the limit $r \to \infty$ below T_g. The possible existence of other parameters was not discussed in Ref. 9.

In the present letter we show that this result is a consequence of an implicit limitation of the class of boundary conditions and that one can introduce an order-parameter function $S(x)$, $x \in [0,1]$.

Following Ref. 9, we write the recurrence relations among the overlap S_i at site i and the overlaps at the sites j which are the nearest sites, external with respect to i, for the case $r \gg 1$; $\overline{J_{ij}} = 0$; $\overline{J_{ij}^2} = J^2/r$:

$$S_i = \iint dz_1 dz_2 \tanh(\beta H + z_1) \tanh(\beta H + z_2) P_i(z_1, z_2, S_{AV}) \equiv \phi(A_{AV}) \tag{1}$$

where

$$P_i(z_1, z_2, S_{AV}) = \frac{1}{2\pi\beta^2 J^2 (q^2 - S_{AV}^2)^{1/2}} \exp\left[- \frac{q(z_1^2 + z_2^2) - 2 z_1 z_2 S_{AV}}{2\beta^2 J^2 (q^2 - S_{AV}^2)} \right], \tag{2}$$

and

$$\beta = \frac{1}{T}; \; q = \langle \tanh^2 U \rangle \equiv \int \frac{dz}{\sqrt{2\pi}} e^{-(z^2/2)} \tanh(\beta H + \beta J z \sqrt{q}) \; S_{AV} = \frac{1}{r} \sum_{J=1}^{r} S_j . \tag{3}$$

The summation in (3) is over all r external neighbors[1)] of site i. In order to determine the overlaps S_i within a tree, we need to specify, in addition to relations (1)–(3), overlap boundary conditions at a remote boundary.

We treat the Bethe lattice as the limit of the Cayley tree of z generations as $z \to \infty$. We number the generations, beginning from the boundary, in such a way that the boundary generation is generation 0, while the central site is generation z. We denote by U_p^k a cluster of r^k boundary sites of such a nature that the common ancestor for any two of them belongs to a generation with a number no greater than k, where $p = 1,2,...,r^{z-k}$. We propose here a hierarchical method for constructing a set of boundary overlaps, which consists of the sequential filling of clusters of progressively greater size: $U_1^1 \subset U_1^2 \subset ... \subset U_1^z$.

•First step. We fill cluster U_1^1 with a set of r overlaps S_i^0, precisely l of which are precisely equal to q, while the others are arbitrary, smaller than q. We denote a cluster filled in this fashion by $U_1^1(l)$ [for example, if we have $S_i^0 = q$ in each, we denote the cluster by $U_1^1(r)$].

•k-th step. We find the cluster $U_1^k(l)$ by combining precisely l clusters $U_1^{k-1}(r)$ and $r-l$ clusters $U_1^{k-1}(l)$:

$$U_1^k(l) = U_1^{k-1}(r) \cup ... \cup U_1^{k-1}(r) \cup U_1^{k-1}(l) \cup ... \cup U_1^{k-1}(l).$$

Proceeding in this fashion, we obtain, in z steps, a complete set of boundary overlaps; i.e., we fill the cluster U_1^z. We see that in this set a self-overlap of two replicas has occurred at precisely $l^z[1-(1-l/r)^z]$ sites; i.e., the overlaps corresponding to these sites are equal to q. It thus seems natural to adopt the ratio $x = l/r$ as the distance between replicas; $x = 1$ corresponds to a complete coincidence of the replicas, and $x = 0$ to their minimal overlap. These two values of x are the values which correspond to the two values of the order parameter q and $\widetilde{S}$ which were found in Ref. 9.

The hierarchical boundary conditions described above lead, along with expressions (1)–(3), to the following equation for the overlap far from a boundary:

$$S(x) = \phi[qx + (1-x)S(x)] \equiv \phi[S_{AV}(x)]. \tag{4}$$

In the limit $r \to \infty$, the ratio $x = l/r$ may be thought of as the argument of a function $S(x)$ which is continuous on $x \in [0,1]$. The stability of the fixed points of the iteration which leads to (4) is determined by the condition $\lambda(x) \leqslant 1$, where

$$\lambda(x) = (1-x)\iint dz_1 dz_2 P(z_1, z_2, S_{AV}(x)) \cosh^{-2}(\beta H + z_1)\cosh^{-2}(\beta H + z_2). \tag{5}$$

It can be seen from (4) that there exists a "replica-symmetric" solution $S(x) \equiv q$. From (5) we find that this solution is stable at $x > x_1$, where

$$x_1 = 1 - (\beta^2 J^2 \langle \cosh^{-4} U \rangle)^{-1}. \tag{6}$$

The condition $x_1 = 0$ determines the transition temperature, which turns out to be the same as the de Almeida–Thouless temperature in the Sherrington-Kirkpatrick model.[9] It is not difficult to find an expression for $S(x)$ at $1 - x/x_1 \ll 1$:

$$S(x) = q - \frac{x_1 - x}{1 - x_1} \frac{\langle \cosh^{-4} U \rangle^2}{2[\langle \cosh^{-4} U \rangle - \langle \cosh^{-6} U \rangle]}. \tag{7}$$

The corresponding eigenvalues satisfies $\lambda(x) = 1 - [(x_1 - x)/(1 - x_1)] < 1$; i.e., this solution is stable. At $x < x_1$ the function $S(x)$ is a strictly increasing function: $dS/dx = \lambda(x)[1-\lambda(x)]^{-1}[q - S(x)](1-x)^{-1} > 0$. We can also write values of the characteristic quantities at $H = 0$, $\tau = 1 - T/J \ll 1$:

$$x_1 = \frac{4}{3}\tau^2 \; ; \; S(0) = 0; \; S(1) = \tau;$$

$$\left.\frac{dS}{dx}\right|_{x=0} = \frac{3}{2\tau} + O(1); \; \left.\frac{dS}{dx}\right|_{x=x_1} = \frac{1}{2\tau} + O(1). \qquad (8)$$

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[1)]It is convenient to assume there that all sites except the central one have $r+1$ neighbors, while the central one has precisely r neighbors.

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