Gauge formulation of theories of relativistic particles and strings

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A principle of the localization of linear canonical symmetries makes it possible to construct relativistic gauge theories of particles and strings from nonrelativistic theories. A relativistic gauge theory of a system of N particles coupled by harmonic forces is constructed. This theory can be quantized by standard methods.

Theories of relativistic particles and strings are usually formulated either in an explicitly reparametrization-invariant form or by means of constraints and Lagrange multipliers. However, neither approach can be adapted easily to describe several inteacting particles or—we wish to stress this point—unusual strings: In the first case, it is necessary to find an invariant Lagrangian (which is nonlinear and which is unrelated to a nonrelativistic Lagrangian). In the second case, the constraints must be specified *a priori*. It was mentioned in Ref. 1 that a relativistic theory of free particles can be constructed from a nonrelativistic localization of linear canonical symmetries of a rudimentary bilinear Lagrangian. In the process one obtains a standard gauge theory in which the canonical variables $y(t)$, $q(t)$, $\xi(t)$ (a Greek letter represents a Grassmann variable) play the role of "fields of matter," and the Lagrange multipliers $l(t)$ appear as components of a gauge potential $A(t)$. In the present letter we show that this simple observation reveals the operation of the fairly general principle of the localization of linear supercanonical symmetries. This principle makes it possible to construct in a unified way not only the known theories but also some absolutely new theories.

Following Refs. 2 and 3, we consider the rudimentary bilinear Lagrangian

$$L_0 = g_{\mu\nu} p_i^\mu \dot{q}_i^\nu - {}^i/_2 h_{\alpha\beta} \xi^\alpha \dot{\xi}^\beta - H_0(p, q, \xi), \tag{1}$$

where $\dot{q} \equiv dq/dt \equiv \partial_t q$; μ, $\nu = 0,1,...,D$; and the index i numbers the particles. The symmetric matrices $g_{\mu\nu}$ and $h_{\alpha\beta}$ can be put in diagonal form. Discarding the variables which contribute vanishing diagonal elements, we find $g_{\mu\nu} = (-1,..., -1, +1,..., +1)$. It can be shown that a systematic quantization of the gauge theory of a single scalar particle which was constructed in Ref. 1 is possible only in the case $g_{00} = -1$, $g_{nn} = +1$. In this sense, relativistic invariance follows from the gauge principle. We will consider only the Minkowski metric below, and we will omit the indices μ and ν. In the choice of Grassmann variables we have a considerable latitude, which we can utilize to describe spin and internal degrees of freedom. For simplicity, we restrict the discussion to the set ξ_k^μ, $k = 1,...,K$; the ξ_k^μ are Lorentz vectors of the same type as p_i^μ, q_i^μ. We introduce the notation $\Psi^T = (p_1, q_1, ..., p_N q_N \xi_1 ... \xi_K)$, and we rewrite L_0 in the form

$$L_0 = {}^1/_2\, \overset{\mathrm{T}}{\Psi} C(\partial_t - H_0)\, \Psi + {}^1/_2\, \partial_t (p_i q_i), \tag{2}$$

where H_0 is a supermatrix of the bilinear form $H_0(p, q, \xi)$. The matrix C can easily be determined; we have $C^2 = -1$. The last term is important for determining the limitations on the parameters of the local transformations $f(t)$, $\varphi(t)$ at the ends of the evolutionary interval $0 \leqslant t \leqslant 1$. That part of L_0 which contains ∂_t is invariant under the global transformations $\delta\Psi = F(f,\varphi)\Psi$ if $F^{\mathrm{T}} C + CF = 0$ [transposition of supermatrices is defined in such a way that we have $(F\Psi)^{\mathrm{T}} = \Psi^{\mathrm{T}} F^{\mathrm{T}}$ when anticommutation is taken into account]; i.e., we have $F \in \mathrm{osp}(2N|K)$. The complete Lagrangian L_0 is invariant under the subalgebra of this superalgebra which is generated by the condition $[F, H_0] = 0$. A Lagrangian which is invariant under local transformations $[F(f(t), \varphi(t))]$ is defined in the standard way:

$$L = {}^1/_2\, \overset{\mathrm{T}}{\Psi} C(\partial_t - A)\, \Psi, \tag{3}$$

where A is found from F by replacing f and φ by the gauge potentials $l(t)$ and $\lambda(t)$. By virtue of the condition $[F, H_0] = 0$, we can usually incorporate the matrix H_0 in A by shifting l and λ by constants. The gauge transformations are also standard:

$$\delta\Psi = F\Psi, \quad \delta A = \dot{F} + [F, A]. \tag{4}$$

In the case of a single particle $(N = 1)$, Lagrangian (3) takes the form

$$L = p\dot{q} - {}^i/_2\, \xi_k \dot{\xi}_k - {}^1/_2\, lp^2 - i\lambda_k (p\xi_k) - {}^i/_2\, \xi_i l_{ik} \xi_k, \tag{5}$$

where $l_{ik} = -l_{ki}$. This Lagrangian describes a theory of massless particles with spin, which was derived in an exceedingly complex way for the case $K = 2$ in Refs. 4 and 5.

Brink *et al.*[4] and Gershin and Tkach[5] also examined a quantization of theory (5) by the Dirac method. The Fradkin-Vilkovisky method[6] is better suited to a systematic, relativistically invariant quantization. In that method, l and λ are treated as new canonical coordinates. Introducing the conjugate momenta k, κ, and adding terms $k\dot{l}$, $\kappa\dot{\lambda}$, to the Lagrangian, we put the gauge potentials on the same footing as that of the main variables. The need for this expansion of the phase space stems from the existence of gauge invariants of the fields l, λ. In the case of a scalar particle, for example, we would have $\delta l = \dot{f}$ and $f(0) = f(1) = 0$, so $\int_0^1 dt\, l(t)$ is gauge-invariant. Consequently, the value of $l(t)$ cannot be fixed by the choice of gauge. An expansion of the phase space by the Fradkin-Vilkovisky method fixes the gauge correctly: $l = 0$. The addition of Faddeev-Popov ghosts to the Lagrangian makes it possible to compensate for the nonphysical degrees of freedom which have been added by virtue of BRST invariance. A standard quantization with a Liouville measure in a phase space expanded in this way leads to the correct expression for the relativistic propagators.[7] By determining fields which depend on the coordinates q^μ, l, and λ and the ghosts, we can find a gauge-invariant formulation of the field theory. This procedure has been carried out for the case of scalar particles by Neveu and West.[8]

For a string, the index i varies continuously. We denoted by the letter s: $0 \leqslant s \leqslant 2\pi$. When we choose the variables $p^\mu(t,s)$, $q^\mu(t,s)$, and $\xi^\mu_\alpha(t,s)$, $\alpha = 1,2$, the simplest rudimentary Lagrangian is

$$L_0 = \int_0^{2\pi} ds\ \mathcal{L}_0(t,s),\ \mathcal{L}_0 = p\dot{q} - {}^1/_2(p^2 + q'^2) + {}^i/_2\,\xi_\alpha\dot{\xi}_\alpha - \xi_\alpha\,\sigma_3^{\alpha\beta}\,\xi'_\beta\,, \tag{6}$$

where $q' \equiv \partial_s q \equiv \partial q$, and σ_3 is a Pauli matrix. Introducing the notation $\Psi^{\rm T} = (p,q,\xi_1,\xi_2)$, we can easily write L_0 in the form of (2), and we can find global symmetries which depend on the parameters $f(s)$ and $\varphi(s)$:

$$\delta\Psi = \mathcal{F}\Psi = D_+ F D_-\,,\ D_\pm = \begin{pmatrix}\partial_\pm & 0\\ 0 & \mathbb{1}\end{pmatrix},\ \partial_+ = \begin{pmatrix}\partial & 0\\ 0 & 1\end{pmatrix},\ \partial_- = \begin{pmatrix}1 & 0\\ 0 & \partial\end{pmatrix},$$

$$F = \begin{pmatrix} f & \varphi\\ \tilde{f} & \tilde{\varphi}\end{pmatrix},\ f = \begin{pmatrix} f_2 & f_1\\ f_1 & f_2\end{pmatrix},\ \tilde{f} = \begin{pmatrix} f_+\partial + \partial f_+ & 0\\ 0 & f_-\partial + \partial f_-\end{pmatrix},\ \varphi = -i\begin{pmatrix}\varphi_1 & -\varphi_2\\ \varphi_1 & \varphi_2\end{pmatrix}, \tag{7}$$

$$\tilde{\varphi} = \begin{pmatrix}\varphi_1 & \varphi_1\\ \varphi_2 & -\varphi_2\end{pmatrix},$$

where $f_\pm = \frac{1}{2}(f_2 \pm f_1)$. The order in which the operators ∂ are applied and the form of a matrix $\tilde{f}$ are determined by the closure conditions, $[\delta_1,\delta_2] \sim \delta_3$. We define $\mathcal{A} = D_+ A D_-$ by means of the replacement $f \to l_1 \varphi \to \lambda$ in $\mathcal{F}$. From (3) we find the Lagrangian

$$\mathcal{L} = p\dot{q} - {}^1/_2\, l_1\,(p^2 + q'^2 + i\,\xi^{\rm T}\sigma_3\,\xi) - l_2\,(pq' + {}^i/_2\,\xi^{\rm T}\xi') - i\,\lambda^{\rm T}\,\xi p - i\,\lambda^{\rm T}\sigma_3\,\xi q'. \tag{8}$$

The gauge transformations are determined by (4), where $A \to \mathcal{A}$, $F \to \mathcal{F}$. This Lagrangian of a fermion string was first found in Ref. 9. Everything said here about the quantization of a theory of particles also applies to a theory of a string. The primary complication is the infinite number of parameters of the gauge group (in a discrete representation, the operator ∂ is determined by the set of matrices $ni\sigma_2$, $n = 1,2,\ldots$). There is the hope that a graphic representation of the gauge structure of the theory will lead to a new approach to the task of constructing a string field theory. If different Grassmann coordinates had been chosen, different string models would result. It would be interesting to list all possible types of gauge string theories.

A rudimentary Lagrangian of N particles coupled by harmonic forces is

$$L_0 = p_i\dot{q}_i - {}^1/_2\,p_i p_i - {}^1/_2\,k_{ij}(q_i - q_j)^2,\ \ i,j = 1,2,\ldots,N. \tag{9}$$

Applying the localization principle, we construct a Lagrangian, which we write explicitly for the case $N = 3$:

$$L = z\dot{y} + z_1\dot{y}_1 + z_2\dot{y}_2 - {}^1/_2\,l_1(z^2 + M^2) - {}^1/_2\,l_2\,(z_1^2 + y_1^2 + z_2^2 + y_2^2 - z^2 - m^2)$$

$$- {}^1/_2\,l_3(z_1^2 + y_1^2 - z_2^2 - y_2^2) - l_4\,(z_1 z_2 + y_1 y_2) - l_5\,(z_1 y_2 - z_2 y_1). \tag{10}$$

Here y, z describe the motion of the center of mass; y_1, z_1 describe the relative motion of the particles (1,2); and y_2, z_2 describe the motion of a third particle in the center-of-mass frame of particles 1,2. The parameter k is cutoff by changing the scale of q and t. The constraints on l_1, l_2 generate translations T_1 and rotations U_1; the constraints on l_3, l_4, l_5 generate the symmetry SU_2. The gauge group is therefore $T_1 \otimes U_1 \otimes SU_2$. This result is generalized to arbitrary values of N. For $N=2$ it is sufficient to set $y_2 = z_2 = 0$ in (10). The Abelian nature of the transformations l_1 and l_2 means that we can add arbitrary mass parameters M^2 and m^2 to (10), but the masses of the individual particles are not determined in this theory. If the k_{ij} are different for the different particles, the SU_{N-1} symmetry is broken. With the appropriate choice of k_{ij} it would be possible to construct a gauge theory of a discrete string. The use of the quantization formalism described above will probably make it possible to construct a relativistic quantum theory for the interaction of composite particles.

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