The fermion string and a universal modulus space

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A way to avoid perturbation theory in the theory of a fermion string is pointed out.

The theory of the fermion string in the Polyakov formalism leads to expressions for a partition function and for string amplitudes in the form of sums of integrals over finite-dimensional superspaces M_p: spaces of the moduli of superconformal manifolds of type p (Ref. 1). An expansion in this sum corresponds to a loop expansion (i.e., a perturbation theory in the topological charge). In order to avoid perturbation theory,

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it is necessary to switch from M_p to a universal modulus space. A corresponding program for a boson string was outlined in Ref. 2, but at this point it is far from completion. We will show here that considerably more progress can be made in the case of a fermion string. In particular, we define a universal modulus space (*UMS*) and construct in it an entity which transforms on each of the M_p into a Mumford superform[3] which may be thought of as a holomorphic square root of a string measure.[4] We then find an expression for the string measure in terms of superanalogs of the Sato τ-function. An analog of the *UMS* may also prove useful in the theory of a boson string. (In the boson case, the manifold *Gr* is used as the universal modulus space[2]; the definition of this manifold is generalized directly to the supercase.[5] However, the manifold *Gr* is wider in both the supercase and the boson case than an universal modulus space should be.)

We start from the superspace H which consists of superfields of the form $F(z,\theta) = f(z) + \varphi(z)\theta$, where $|z| = 1$, and θ is an odd variable. We define a bilinear scalar product in H by means of $\langle F,F'\rangle = \oint f(z)f'(z)dz + \oint \varphi(z)\varphi'(z)dz$, where $|z| = 1$. We denote by S a subset of the basis $\{z^k,z^k\theta\}$, where k is an integer, which is found by connecting and removing a finite number of elements from the S_0 subset $\{z^k,z^k\theta,k<0\}$. We denote by $H(S)$ a subspace constructed on S. We denote by *Gr* a supermanifold which consists of subspaces $W\subset H$ for which the projection onto one of the $H(S)$ is an isomorphism. The group Γ of reversible, even, smooth superfields $A(z,\theta = a(z) + \alpha(z)\theta$, $|z| = 1$, acts in H and in *Gr* with the help of multiplication operators. We denote by *UMS* the submanifold of the set *Gr* which consists of those $W\in Gr$ with $A\in\Gamma$ for which we have $W^\perp = \Pi A W$ [$W^\perp$ is the orthogonal complement of W; the operator Π transforms $f(z) + \varphi(z)\theta$ into $\varphi(z) + f(z)\theta$]. We denote by N a complex compact supermanifold of dimensionality (1, 1); L is a linear stratification with a base N. If we choose a coordinate system (z,θ) near one of the points N, and if we fix the trivialization of the stratification L on this neighborhood, we determine $W = W(N,L)$ as the space of fields $f(z) + \varphi(z)\theta$, $|z| = 1$, which continue into holomorphic cross sections above $N\setminus U_0$, where U_0 is distinguished by the inequality $|z| < 1$. It is easy to see that we have $W(N,L)\in Gr$ and $AW(N,L) = W(N,L')$, where $A\in\Gamma$, and L' is another linear stratification on N. It can be shown that we have $W(N,L)\in UMS$. The proof is based on the relation $W(N,L)^\perp = \Pi W(N,L^{-1}\otimes\omega)$, where ω is a stratification whose cross sections are complex measures on N. We can thus interpret *UMS* as a universal modulus space, and we can treat the points of *UMS* as supermanifolds of an arbitrary (possibly infinite) kind. We consider finite-dimensional superspaces $\Sigma(W) = \mathscr{A}(W) + \mathscr{A}(W^\perp)$, where $\mathscr{A}(W)$ consists of elements of the space W which vanish when W is projected onto $H(S_0)$. With any basis w in $\Sigma(W)$ we can associate a basis $\hat{w}$ in W, determined within a transformation having a unit berezinian. [For example, if $\mathscr{A}(W^\perp) = 0$, a basis $\hat{w}$ can be constructed by complementing the basis w in $\Sigma(W) = \mathscr{A}(W)$ with functions which are mapped upon a projection onto $H(S_0)$ into the basis S_0 of the space $H(S_0)$.] Denoting by w a basis in $\Sigma(W)$ and by w' a basis in $\Sigma(AW)$, we determine $\tau(w,w',W,A)$ as the determinant of the transformation from the basis $A\hat{w}$ in AW to the basis $\hat{w}'$. (In the supercase, in contrast with the boson case, this determinant is finite.) Using the function τ, we can interpret the Mumford superform[3] in terms of a *UMS*. Specifically, if we have

$W \in UMS$, $W^{\perp} = \Pi A W$, $A \in \Gamma$, we set $M(w,w',W) = \tau(w,w',W,A^3) \times \tau(w,\Pi w,W,A)^{-3}$; here w is a basis of $\Sigma(W) = \Sigma(W^{\perp})$, Πw is a basis of $\Pi\Sigma(W^{\perp}) = \Sigma(\Pi W^{\perp}) = \Sigma(AW)$, and w' is a basis of $\Sigma(A^3 W)$. The function $M(w,w',W)$ does not depend on the choice of the operator $A \in \Gamma$; it has a weight of 1 on w' and a weight of 5 on w. [*A function of the basis has a weight n if it is multiplied by (Ber C)ⁿ upon* the replacement of the basis with matrix C.] If $W = W(N,L)$, where N is a superconformal manifold, and L is a trivial stratification, then $\Sigma(A^n W)$ is identified with $\mathscr{A}^{n/2} + \Pi\mathscr{A}^{(1-n)/2}$, where $\mathscr{A}^k$ is the space of holomorphic fields of type $(k,0)$ on N, and Π is the parity-reversal operator. For this identification, $M(w,w',W)$ transforms into a Mumford superform. The function M determines a continuation of a Mumford superform into an entity which is defined on the space of moduli of all (not necessarily superconformal) (1, 1)-dimensional compact supermanifolds. The possibility of such a continuation also follows from Ref. 3. For any point $W = W(N,L) \in UMS$ corresponding to a superconformal manifold N one can also find functions P and Q of z, θ, $|z| = 1$, such that the operator $PD + Q$ sends W into itself (as usual, $D = \partial/\partial\theta + \theta\partial/\partial z$). We have found an expression for the string measure in terms of the function τ (Ref. 4). This function is an analog of the τ-function in the Sato sense, but its properties are simpler. Specifically, the dependence of the function τ on A with $W = W(N,L)$ can be described by means of meromorphic Abelian integrals on supermanifold N.

We conclude by showing how the UMS is described in terms of a Fok space, restricting the discussion to the boson case for simplicity. In this case, the role of H is played by a space of functions on the circle $|z| = 1$ with the scalar product $\oint f(z)f(z)dz$, $|z| = 1$. On the complex curve N and the stratification L we construct a subspace[2] $W = W(N,L) \in Gr$. These subspaces satisfy the condition $W^{\perp} = AW$, where A is the operator representing multiplication by a function. By choosing L appropriately we can achieve $A \equiv 1$ (for this case, we should treat L as a spinor stratification with an even nondegenerate θ-characteristic). Accordingly, the UMS in the boson case should be determined as a manifold of subspaces $W \in Gr$ which satisfy the condition $W^{\perp} = W$. With each $W \in UMS$ we can associate a vector Φ, determined within a coefficient, from the fermion Fock space with the annihilation operator a_n and the creation operator a_n^+, $n = 0,1,2,\ldots$, finding it from the relation $(\oint f(z)\psi(z)dz)\Phi = 0$, where $f \in W$ and $\psi(z) = \Sigma a_z z^n + \Sigma a_n^+ z^{-1-n}$, $n \geqslant 0$. Conversely, if the vector Φ can be found from a Fock vacuum by means of a linear canonical transformation $\tilde{a}_n = \Sigma u_{nk} a_k + \Sigma v_{nk} a^+{}_k$, then Φ corresponds to some $W \in UMS$. Corresponding constructions can be pointed out in the supercase.

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[1]M. A. Baranov and A. S. Shvarts, Pis'ma Zh. Eksp. Teor. Fiz. **42**, 340 (1985) [JETP Lett. **42**, 419 (1985)].

[2]D. Friedan and S. Shenker, Phys. Lett. **B175**, 287 (1986); Yu. I. Manin, Funkts. Anal. Pril. **20**, 88 (1986); L. Alvarez-Gaume, C. Gomez, and C. Reina, Phys. Lett. **B190**, 55 (1987); A. Yu. Morozov, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 457 (1987) [JETP Lett. **45**, 585 (1987)].

[3]A. A. Voronov, Funkts. Anal. Pril. **21** (1987).
[4]M. A. Baranov and A. S. Schwarz, Int. J. Mod. Phys. **A2**, (1987).
[5]Yu. Manin and A. Radul, Commun. Math. Phys. **98**, 65 (1986); K. Ueno and H. Yamada, Lett. Math. Phys. **13**, 59 (1987).

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