Cellular structure of space near a singularity in time in Einstein's equations

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It is shown on the basis of one of the general cosmological solutions of Belinskiĭ and Khalatnikov {Zh. Eksp. Teor. Fiz. **63**, 1121 (1972) [Sov. Phys. JETP **36**, 591 (1972)]} that as a metric evolves toward a singularity in a stochastic oscillatory regime, some regions with a small-scale cellular 3-geometry structure form near it. The statistics of the exponents and the time evolution of lengths in such regions are studied.

1. An oscillatory regime near an isolated spatial point in relativistic cosmology has been studied in detail.[2,3,1] In the present letter we examine the effect of a prolonged oscillatory evolution of the metric on the structure of the exponents in a coordinate-finite spatial region near a singularity. For this purpose we use the example of a nonvacuum general solution whose evolution includes an oscillatory stage and a monotonic stage (containing a singularity), as constructed by Belinskiĭ and Khalatnikov: In an individual epoch we have $ds^2 = dt^2 - \Sigma(t^{p_l(x)} l_\alpha(x) dx^\alpha)^2$, $\Sigma p_l(x) = 1$, $\Sigma p_l^2(x) = 1 - q^2(x)$ [$t \ll 1, x \equiv (x^\alpha)$, where $\alpha = 1, 2, 3$; all the sums are over the three axes l, m, n; and $q = 0$ corresponds to the well-known Belinskiĭ-Lifshitz-Khalatnikov solution[2]]. The oscillatory stage combines epochs with a negative exponent in the metric. In the given epoch and in the epoch which follows it (in the direction $t \to 0$), the exponents are coupled by a local mapping: For $p_l < 0$,

$$p_l' = \frac{-p_l}{1+2p_l}, \quad p_m' = \frac{p_m + 2p_l}{1+2p_l}, \quad p_n' = \frac{p_n + 2p_l}{1+2p_l}, \quad q' = \frac{q}{1+2p_l}. \tag{1}$$

Iterations of (1) lead to an epoch with $p_l(x), p_m(x), p_n(x) > 0$, which lasts until $t = 0$ (the monotonic stage at point x). The reasons for choosing this solution are that (1)

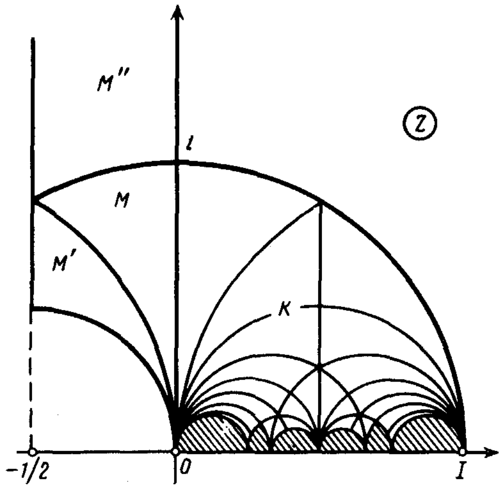

FIG. 1. Part of the plane of the complex parameter $(-p_1+iq/\sqrt{2})/(1-p_2)=z$ with $\operatorname{Im} z\geqslant 0$ (the pattern is the same at $\operatorname{Im} z\leqslant 0$). Heavy lines—the boundaries of K, M, M', M'', $z\in K$: $p_1<0<p_2\leqslant p_3$, $z\in M$: $0\leqslant p_1\leqslant p_2\leqslant p_3$, $z\in M'$: $0\leqslant p_2\leqslant p_1\leqslant p_3$, $z\in M''$: $0\leqslant p_1\leqslant p_3\leqslant p_2$. Several subregions K_r in K are shown; the others are hatched.

the geometry in it is richer than in the Belinskiĭ-Lifshitz-Khalatnikov solution, since the metric contains an additional arbitrary function $q(x)$, (2) the transition to the monotonic stage makes it possible to describe the spatial structure of the exponents in a simple way near $t=0$, and (3) in the class of nearly vacuum initial conditions these exponents have a quasi-invariant statistical distribution in the monotonic stage [oscillatory stage: $t=t_1$:$q^2(x)\ll 1$] by virtue of the onset of a stochastic situation in the oscillatory stage.

2. Using the ordered indices $p_1\leqslant p_2\leqslant p_3$, we introduce the definition

$$\frac{-p_1+i\kappa q/\sqrt{2}}{1-p_2}\equiv\begin{cases} u+iv=w\in K: p_1(u,v)<0 & \text{(OS)}\\ U+iV=W\in M: p_1(U,V)\geqslant 0 & \text{(MS)}\end{cases}\tag{2}$$

(see Fig. 1 regarding the regions K and M and also the regions M', M'', and K_r, which are introduced below). The quantity $\kappa=\pm 1$ transforms as $\kappa'=-\kappa\,\operatorname{sign}[(p_2+3p_1)(p_3+3p_1)]$ upon a change of epochs. From (1) and (2) we have a chain fraction $(a_1,\ldots,a_n=1,2,\ldots)$ for $W(w)$:

$$\begin{gathered}W(w)=\delta_{1J}\,z(w)-\delta_{2J}\,[1,z(w)]+\delta_{3J}/z(w),\\ z(w):\ \{w=[a_1,\ldots,a_n,z],\quad w\in K, z\in M\cup M'\cup M''\},\end{gathered}\tag{3}$$

where $J=1, 2, 3$, respectively, for $z\in M$, M', M''. Here $W(w)$ is a continuous, piecewise-conformal mapping of K onto M with a countable set of inverse images K_r of the region M in K, and we are using the "multi-index" $r=\{\alpha_1,\ldots,\alpha_n;J\}$. In the limit

Im $w \to 0$, it has a mixing (similar to a Gauss mapping[4]) and an invariant metric in M [see (5)]. In the oscillatory stage we specify a function $w(x)$ (assuming that the change of epochs occurs instantaneously at $t = t_1$) in the 3-space region A. As $t \to 0$ (in the monotonic stage), we denote the same region by B. For the mapping $w(x):A \to K(A) \subset K$ the partitioning $K = \cup_r K_r$ corresponds to $A = \cup_{r'} A_{r'}$, $\{r'\} \subset \{r\}$. For simplicity, we assume $K(A) = K$ and that all the A_r are singly connected. Two commuting pairs of single-valued mappings, $w(x):A_r \to K_r, W_r(w):K_r \to M$ and $t(t_1) \to 0:A_r \to B_r, W[w(x)]:B_r \to M$, determine a partitioning of the region $B = \cup_r B_r$ into a countable set of cells, in each of which the set of exponents $(p_1\ p_2\ p_3)$ takes on all the values allowed algebraically under the condition $p_1 \geqslant 0$. Over n epochs, $\sim 2^n$ cells form. The boundary of a cell is a two-dimensional surface of a degenerate dynamics: It consists of three pieces, on which we have either $p_1 = 0$ or $p_1 = p_2$ or $p_2 = p_3$. The pieces are joined transversally. The cells are "glued" to each other along corresponding pieces. It is easy to show that the surface of a cell cannot be homeomorphic with respect to a sphere, and in the typical case it is topologically a torus or a cylinder. Cells with $g \geqslant 2$ handles are atypical: They arise only if there are points with $\widehat{\nabla} U(x) = 0$ or $\widehat{\nabla} V(x) = 0$ at their boundaries.

We now assume that $w(x)$ corresponds to a small q-perturbation of the Belinskiĭ-Lifshitz-Khalatnikov solution in A. In other words, we assume that the distribution in u, v in A at $t = t_1$,

$$\rho_A(u, v; [w(x)]) = \Omega^{-1}(A) \int_A \delta(w(x) - w) \sqrt{\gamma}\, d^3x, \quad \Omega(A) \equiv \int_A \sqrt{\gamma}\, d^3x \tag{4}$$

[a functional of $w(x)$], is of such a nature that we have $|\bar{v}| \ll \sigma v \ll 1$ and $\epsilon \equiv \sigma v / \sigma u \ll 1$ ($\bar{v}$ and $\sigma^2 v$ are the mean value and the variance). Under reasonable restrictions on the form of ρ_A, a lower estimate of the number of cells in B with a total volume $\sim \Omega(B)$ is $N \sim \epsilon^{-1} \gg 1$; i.e., $0 < \Omega(B_r) \lesssim \epsilon \Omega(B)$. A set of cells forms a small-scale geometric "foam." Only in the limit $\epsilon \to 0$, in which we have $t_* \to 0$ (the time of the transition to the monotonic stage), is a solution with $p_1 = 0$ stable, $t < t_*$ [if, on the other hand, t_* is not zero, then on the interval $0 < t < t_*$ a drift will arise from $p_1(t_*) = 0$ to some $0 < p_1(0) \ll 1$].

The stochastic behavior in (3) in the limit Im $w \to 0$ generates a universal statistics for U, V in region B as $\epsilon \to 0$: On any family $\{w_\epsilon(x)\}$ there exists $\lim_{\epsilon \to 0} \rho_B(U, V; [W(w_\epsilon(x))]) \equiv \rho(U, V)$. Analysis of (3) yields

$$\rho(U, V) = \frac{2}{\pi V^2} \Big[3 - \frac{U}{V} \arctan \frac{V}{U} - \frac{U+1}{V} \arctan \frac{V}{U+1} - \frac{U+U^2+V^2}{V} \arctan \frac{V}{U+U^2+V^2} \Big]. \tag{5}$$

From (5) we find the numerical results $(\bar{p}_1; \bar{p}_2; \bar{p}_3; |\bar{q}|) = (0.06; 0.17; 0.77; 0.50)$.

3. The length of the vector λ^α on the set of realizations $W(x)$ is a random quantity $[\Sigma(t^{p_l} l_\alpha \lambda^\alpha)^2]^{1/2} \equiv \lambda(t)$. Its moment $\overline{\lambda^s}$ $(s > 0)$ decreases as $t \to 0$ as the Laplace integral $\int t^{sp_1} \rho(p_1) \varphi(p_1) dp_1$, where $\rho(p_1)$ is the transformed distribution (5). Here we have $\rho(p_1 \ll 1) \sim p_1^{-1/2}$, and $\varphi(p_1)$ is a contribution of the correlation of the vectors l_α,

m_α, n_{α_2} with W, which arises because of their rotations in the oscillatory stage. The nonlocal nature of the rotation formulas[2] prevents us from finding φ, but we can show that we have $0<\varphi(0)<\infty$. Consequently, we have $\lambda^s(t\to 0)\sim(-\ln t)^{-1/2}$; with regard to the local length, we can only assert that it falls off slowly as $t\to 0$. From inequalities of the Chebyshev type it follows that the probability satisfies

$$P\ \{\lambda(t)\geqslant e\tau^{-1/2}\ln L(\tau)\}\leqslant 1/L(\tau),\quad \tau\equiv-\ln t\to\infty, \tag{6}$$

where $L(\tau)$ is an arbitrary slow-growth function: As $\tau\to\infty$ and with $c>0$ we have $L(c\tau)/L(\tau)\to 1$; i.e., (6) is invariant under the replacement $t\to t^c$.

We now assume that $\lambda^\alpha(\theta)$ is a tangent vector of a curve of general position $\mathbb{C}$: $\{x^\alpha=x^\alpha(\theta),\ \theta\in[a,b]\}$. Partitioning $[a,b]$ into N equal parts Δ_k, we find the length of the curve to be $l_{\mathbb{C}}(t)=\int_a^b d\theta\lambda(t,\theta)=N^{-1}\Sigma_k\lambda_k(t)$, where $\lambda_k(t)\equiv\lambda(t,\theta_k\in\Delta_k)$. In the limit $\epsilon\to 0$, the $\lambda_1,\ldots,\lambda_N$ give us N independent random quantities, and in the limit $N\to\infty$ we have $\epsilon N\to 0$ with a probability of 1:

$$l_{\mathbb{C}}(t)\sim(-\ln t)^{-1/2},\quad t\to 0. \tag{7}$$

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