Percolation models with disorder

S. P. Obukhov
L. D. Landau Institute of Theoretical Physics, Academy of Sciences of the USSR

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The critical indices in the first $4-\epsilon$ approximation are calculated for the problem of directed percolation with disorder. The percolation transition is an infinite-order transition for disordered layered systems.

Percolation models are customarily used to describe the properties of conductivity, percolation, and connectedness in disordered systems. An extreme local disorder is built into these models from the outset (in the percolation problem of bonds, e.g., this would be the presence or absence of bonds between adjacent lattice sites). If an additional disorder is introduced in such a system (if the bond problem is solved on a lattice with defects), the only change will be in the threshold for percolation in this system—not in its critical properties near the threshold. Simple estimates show that the disorder must be nonlocal in order to change the universality class of a percolation problem.

A percolation problem can also be used for a space-time description of random processes in excited media (the so-called problem of directed percolation[1]; see the reviews in Refs. 2–4). In real biological media, neuron networks, and logic networks which can be described by a directed-percolation problem, there are always defects and a spatial inhomogeneity, which remain unaltered throughout the evolution of the process. In a space-time description of a system with point defects, the theory is nonlocal in time. Noest[5] has shown by numerical simulation that the critical indices of a system with defects are quite different from the indices of a homogeneous problem.

Let us consider the very simple model in which each site in a d-dimensional space lattice can be in two different states: the ground state and an excited state. If site i is excited at time t, it may at the following time, $t+1$, put its nearest neighbors in an excited state, with an independent probability p_i for each neighbor; with the same probability it may remain excited itself. We assume that the inhomogeneity of the system is manifested in an inhomogeneity of p_i: $p_i = p + \Delta_i$, where $\langle \Delta_i \rangle = 0$,

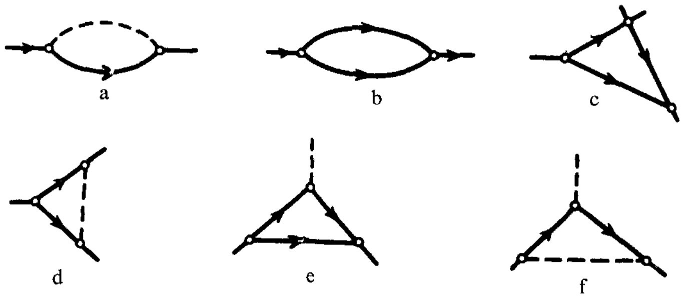

FIG. 1.

$\langle \Delta_i \Delta_j \rangle = s^2 \delta_{ij}$, and p is the mean value of p_i. If p exceeds the critical value p_c, the excitation of one site can lead to a stochastic process which does not decay over time. The description of such a process reduces to a percolation problem on a $(d+1)$-dimensional space-time lattice, where the sites are connected by random bonds, which are oriented in the direction of increasing time. The inhomogeneity of the distribution of random bonds can be taken into account by using the diagram technique which was developed for the homogeneous problem in Ref. 1.

Figure 1a shows the simplest correction for an inhomogeneity to the correlation function of a percolation problem. With the solid lines we associate nucleating correlation functions of the directed-percolation problem, which are of the form $G(k,\omega) = (k^2 - i\omega + \tau)^{-1}$, where $\tau \propto p_c - p$, in the momentum representation; with a dashed line we associate a factor $\sigma(\omega)/2\pi$; and with a vertex representing an interaction with an impurity we associate a factor s. In the limit $\tau \to 0$, the correction in Fig. 1a has precisely the same singularity as the first correction to $G(k,\omega)$ in the homogeneous problem (Fig. 1b; see also Ref. 1). In the loop in the diagram in Fig. 1b, there is one more correlation function, but an additional integration over the frequency must be carried out here. This additional integration is not part of the preceding correction, because of the factor $\delta(\omega)$. Analysis of more-complex diagrams shows that incorporating them reduces to introducing a renormalized triple vertex (the simplest corrections are shown in Figs. 1c and 1d) and a renormalized vertex representing the interaction with the impurity (Figs. 1e and 1f).

The renormalization-group equations for both renormalized vertices, $\tilde{g}$ and $\tilde{s}$, are

$$\frac{\partial \tilde{g}}{\partial \xi} = \tilde{g}\left\{\frac{1}{4}(\epsilon - 2\eta_\perp - \eta_\parallel) - \frac{k_4}{4}\tilde{g}^2 + \frac{3k_4}{2}\tilde{s}^2\right\}$$

$$\frac{\partial \tilde{s}}{\partial \xi} = \tilde{s}\left\{\frac{1}{4}(\epsilon - 2\eta_\perp) - \frac{k_4}{8}\tilde{g}^2 + \frac{k_4}{2}\tilde{s}^2\right\}, \qquad (1)$$

where $\eta_\parallel = -k_4(\tilde{g}^2/8) + k_4 s^2$, $\eta_\perp = -(k_4\tilde{g}^2/16)$, $k_4 = (32\pi^2)^{-1}$, $\xi = -\ln|p_c - p|$, $\epsilon = 4 - d$.

Equations (1) have a stable fixed point:

$$\tilde{g}_f^2 = 8k_4^{-1}\epsilon , \quad \tilde{s}_f^2 = k_4^{-1}\epsilon , \qquad (\tilde{g}_f^2 = \frac{4}{3}k_4^{-1}\epsilon) . \tag{2}$$

Here and below, quantities characterizing the homogeneous directed-percolation problem are enclosed in parentheses ($s \equiv 0$).[1]

Finally, we write the expressions for the primary critical indices of the problem of inhomogeneous directed percolation: the indices of the longitudinal and transverse correlation length, the density index of an infinite cluster, and the susceptibility index (the mean size of a cluster):

$$\begin{aligned} &\nu_{\|} = 1 + \frac{\epsilon}{2}, \quad \nu_{\perp} = \frac{1}{2} + \frac{\epsilon}{8}, \quad \beta = 1 + O(\epsilon^2), \quad \gamma = 1 + \frac{\epsilon}{2}, \\ &\left(\nu_{\|} = 1 + \frac{\epsilon}{4}, \quad \nu_{\perp} = \frac{1}{2} + \frac{\epsilon}{16}, \quad \beta = 1 - \frac{\epsilon}{6}, \quad \gamma = 1 + \frac{\epsilon}{6}\right). \end{aligned} \tag{3}$$

The quantity $\tilde{s}^2$ appears in Eq. (1) only with a plus sign; if the coefficient of $\tilde{s}^2$ were slightly larger, system (1) would be unstable. This proximity to instability is manifested by a stronger dependence of $\tilde{g}_f^2$ and of the critical indices on ϵ than in the homogeneous problem. This situation agrees qualitatively with the anomalously large values of the critical indices which were found in a numerical simulation of the inhomogeneous problem[5] in dimensionalities $d = 1$ and $d = 2$. A quantitative agreement between the critical indices in (3) and experimental values would probably be possible only at $d = 3$.

We conclude with a brief discussion of the role played by a nonlocal disorder in the ordinary percolation problem with nondirectional bonds. For this purpose, we compare the corrections in Figs. 1a and 1b for the case of ordinary percolation with the nucleating correlation function $G(k) = (k^2 + \tau)^{-1}$. If the contribution from the diagram in Fig. 1b is to be comparable in magnitude with that from the diagram in Fig. 1a in the limit $\tau \to 0$, the correlation function for the disorder must be of the form $1/k^2$, or it must contain δ-functions, which reduce the number of integrations in the loop in the diagram in Fig. 1b by two. An example illustrating this second possibility is a layered system, in which the density of random bonds varies randomly from layer to layer. This type of nonlocal disorder absolutely must be considered in models describing the permeability of stratified geological objects, e.g., petroleum and gas deposits, or in models describing the properties of layered composite materials.

Remarkably, the greatest contribution to the measurable values from an inhomogeneity in a layered system does not arise in perturbation theory and instead is of an "instanton" nature. This contribution arises from the exponentially rare regions of the system, in which there are several layers in a row in which the density of random bonds is higher than the mean density sufficient for the formation of two-dimensional percolation along these layers. Let us assume, for example, that the mean density of bonds, p, is lower than p_c but that the density of bonds in each layer varies at random as $p \pm \Delta$ and $p + \Delta > p_c$ (p_c is the critical density of bonds in the homogeneous system). If two-dimensional percolation is to arise, there must be several layers in a row with an elevated bond density $p + \Delta$. The thickness of these stacks must be on the order of the correlation radius for the homogeneous percolation problem: $r_c \sim (p+\Delta-p_c)^{-\nu}$, where $\nu \simeq 0.8$ for $d=3$. At $r_c \gg 1$, a random stacking of layers of this sort would be exponentially rare, $\sim \exp(-|p+\Delta-p_c|^{-\nu})$. In an infinite system, the percolation transition occurs at the point $p=p_c-\Delta$ [not at the point $p=p_c+0(\Delta)^2$, as it would according to perturbation theory]. At this point, all the mean measurable quantities are infinitely differential (e.g., the density of an infinite cluster). This transition is thus not of second order, as in the homogeneous percolation problem, but of infinite order.

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