An inequality for the density of condensate atoms in superfluid ^{4}He

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An inequality setting an upper limit on the density of the condensate of a superfluid Bose fluid at $T = 0$ is derived. In the derivation advantage was taken of the fact that the excited states with $\mathbf{k} \neq 0$ are separated from the ground state by a finite gap. A numerical calculation for ^{4}He yields $n_0/n < 0.77$.

Of all the quantities characterizing the superfluidity of liquid ^{4}He the density of the condensate particles, n_0, has so far been studied least of all. The data on the scattering of neutrons in helium give values ranging between 3% and 35% for the ratio n_0/n (Ref. 1–3). The interpretation of the relevant measurements is not, however, completely unambiguous. As regards the theory, there is no general equation which is similar to the Landau equation for the density of the superfluid part, ρ_s, which can express n_0 in terms of other observable quantities such as the elementary-excitation spectrum. In Refs. 4 and 5 values on the order of 10% were obtained for n_0/n using various approximations. It is difficult to tell, however, to what extent these estimates are reliable.

We will derive in a rigorous manner an inequality which sets an upper limit on n_0. The boundary is determined by the type of excitation spectrum.

The derivation is based on the fact that at absolute zero temperature the Bose fluid, which has at small $\mathbf{k}$ a phonon energy spectrum, has at any finite $\mathbf{k}$ a certain minimum energy $\epsilon_m(k) > 0$, which can be imparted to the liquid. The quantity $\epsilon_m(k)$ is equal to the excitation energy $\epsilon(k)$ at those values of $\mathbf{k}$ at which there are undamped elementary excitations in the liquid (excitations incapable of decay). At other values of $\mathbf{k}$, $\epsilon_m(k)$ is equal to the minimum energy of several excitations with a total momentum $\mathbf{k}$. This energy, which is clearly finite, can be determined if the behavior of the spectrum in the "undamped" parts is known. Note that the Landau superfluidity condition is satisfied because of the finite nature of $\epsilon_m(k)$. For a spectrum of the type $\epsilon(k) = \hbar^2 k^2/2m^*$, which does not satisfy this condition, we have $\epsilon_m(k) \equiv 0$.

In an actual ^{4}He at zero pressure in the region $k \lesssim 0.6$ Å^{-1}, where the phonons can decay, we have $\epsilon_m(k) = \hbar k u$ (u is the speed of sound). At higher values of k, $\epsilon_m(k)$ is equal to $\epsilon(k)$ to within $k = k_c \approx 2.7$ Å^{-1}, where $\epsilon(k)$ reaches the threshold value 2Δ. At $k_c < k < 2k_0$, where the excitations can decay into two rotons, we have $\epsilon_m(k) = 2\Delta$. (Δ and k_0 are the energy and momentum of the roton minimum.) In the limit $k \to \infty$, $\epsilon_m(k)$ is probably equal to the energy of a circular vortex ring with the momentum k.

We will make use of the Bogolyubov inequality[6] for $A(\omega,k)$—the space-time Fourier transform of the mean value of the commutator of the creation and annihilation operators of the atoms of the liquid:

$$A(\omega, k) = \langle [\psi(t, \mathbf{r}), \psi^+(0, 0)] \rangle_{\omega \mathbf{k}} \, .$$

The sign of A is the same as the sign of ω. The momentum distribution function of the particles above the condensate is expressed in terms of this quantity (A):

$$N_{\mathbf{k}} = - \int_{-\infty}^{0} A(\omega, k) \frac{d\omega}{2\pi} \, , \qquad T = 0. \tag{1}$$

The sum rule and the inequality

$$\int_{-\infty}^{\infty} A(\omega, k) \frac{d\omega}{2\pi} = 1 , \qquad \int_{-\infty}^{\infty} \frac{A(\omega, k)}{\omega} \frac{d\omega}{2\pi} \geqslant \frac{n_0 m}{n \hbar k^2} \tag{2}$$

are valid for $A(\omega,k)$ at all k (m is the mass of the helium atom; see Refs. 6–8).

Because of the finite value of $\epsilon_m(k)$, the integration in (1) and (2) is actually taken over only the region $|\omega| \geqslant \epsilon_m/\hbar$. This enables us to derive from (2) an inequality for N_k, as it was done in Ref. 6 for small k.

We can accordingly rewrite the left side of inequality (2) in the form

$$\int_{\epsilon_m/\hbar}^{\infty} \frac{A(\omega, k) - A(-\omega, k)}{\omega} \frac{d\omega}{2\pi} \, . \tag{3}$$

The integrand in (3) is positive and the integral will increase if ω in the denominator is replaced by the minimum value of $\epsilon_m(k)/\hbar$. We can then infer from (1) and (2) that

$$N_{\mathbf{k}} \geqslant \frac{n_0 m \, \epsilon_m(k)}{2n \hbar^2 k^2} - \frac{1}{2} \, . \tag{4}$$

Inequality (4) is valid for all $\mathbf{k}$, although it has a real meaning only in the region where the right side is positive. Let us now integrate (4) over the sphere in k space with a radius Λ which is arbitrary for the time being. Clearly,

$$n - n_0 = \int N_{\mathbf{k}} \frac{d^3k}{(2\pi)^3} > \int_{k<\Lambda} N_{\mathbf{k}} \frac{d^3k}{(2\pi)^3}$$

Solving the inequality for n_0/n, we find

$$\frac{n_0}{n} < \delta(\Lambda) = \frac{1 + \dfrac{\Lambda^3}{12\pi^2 n}}{1 + \dfrac{m}{4\pi^2 n \hbar^2} \int_0^\Lambda \epsilon_m(k) dk} \quad (5)$$

We thus see that for a spectrum which is linear for small k, $\delta(\Lambda)$ has a minimum of less than unity at a certain Λ_m. The strongest inequality for n_0/n that can be obtained in this manner comes from the corresponding value of δ_m. The value of δ_m for superfluid ^{4}He was calculated from the experimental data of Ref. 9 on the excitation spectrum. The calculations gave $\Lambda_m \approx 0.92$ Å^{-1} (which is near the maximum of the spectrum) and $\delta_m \approx 0.77$ at zero pressure. At a pressure of about 25 atm $\delta_m \approx 0.7$. Note that in the calculations we used only those values of $\epsilon_m(k)$ where $k \leqslant \Lambda_m$, so that these values differ from $\epsilon(k)$ only slightly.

The maximum value of n_0/n obtained by us is probably much higher than the true value. The fact that this quantity can be rigorously estimated is nonetheless of considerable practical value.

Inequality (5) can be used for the nonideal Bose-gas model. Since the energy spectrum found by Bogolyubov is a "decay"-type spectrum, we have $\epsilon_m(k) = \hbar k u$. Calculation of the right side of (5) yields the inequality $(1 - n_0/n) > (1/24\pi^2 n)(mu/\hbar)$, whereas according to Ref. 10, we have $(1 - n_0/n) = (1/3\pi^2 n)(mu/\hbar)^3$.

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