Partition functions for open and/or oriented strings

A. Morozov and A. Roslyĭ
Institute of Theoretical and Experimental Physics

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A mapping of the spaces of the moduli of open and/or nonorientable Riemann surfaces into spaces of the moduli of closed orientable surfaces is described. In this mapping, the components of the partition functions and of the amplitudes of boson strings are expressed in terms of the Mumford measure.

In a first-quantized string theory in the critical space-time dimensionality, measures of an integration on a universal space of moduli of Riemann surfaces are associated with all physical quantities. The components of this space—the spaces of moduli of surfaces with a given topology—are important for a loop expansion. Closed unoriented boson strings correspond to closed oriented Riemann surfaces without an

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edge. Such surfaces have a unique topological characteristic: the type p, which is a natural number (the number of handles). The corresponding space of moduli M_p is complex, and the measure of integration on it, which determines the p-loop component of the partition function of the strings, is[1]

$$dv_p \simeq |d\mu^{(p)}_{bos}|^2/(\det \mathrm{Im}\, T)^{13\ (14)} \ . \tag{1}$$

The expressions for other string theories can also be conveniently reduced to certain expressions which are defined on M_p, although the natural spaces of moduli that arise in these problems are not identical to M_p. For superstrings and heterotic strings, for example, the starting point is a space of supermoduli, and the measure on M_p arises only after an integration over the odd generators of this space.

In this letter we are interested in the partition functions (vacuum diagrams) of open and/or oriented strings. A new element here is the appearance of components of nonorientable Riemann surfaces and/or surfaces with an edge. Their topology is characterized by three non-negative integers: the number of handles, h, the number of holes, m, and the number of holes sealed by Möbius sheets, c. A surface is orientable only in the case $c=0$. The spaces of moduli of such surfaces, $M_{h,c,m}$ with $c,m\neq 0(M_{p,00}=M_p)$, are not complex, and their real dimensionality is $\dim_R M_{h,c,m} = 6h+3c+3m-6+\rho_{h,c,m}$. Here $\rho_{h,c,m}$ is the dimensionality of the continuous group of conformal mappings of the surface onto itself; here we have $\rho\neq 0$ only for a disk, a sphere, a projective space, a ring, a torus, a Klein bottle, and a Möbius sheet.

The measures on the spaces $M_{h,c,m}$ which correspond to string partition functions take a simple form after $M_{h,c,m}$ are nested into the spaces M_p with $p=2h+c+m-1$ as real spaces of half the dimensionality. This nesting is described in terms of doubles of Riemann surfaces, and the measure is expressed in terms of the Mumford measure $d\mu^{(p)}_{\mathrm{bos}}$.

With each open and/or nonorientable Riemann surface S one associates a "double," D: a closed orientable surface with an antiholomorphic Z_2 isometry, so that we have[2] $S=D/Z_2$. (For a disk and a projective plane, for example, the RP^2 double is a sphere; for a ring, a Klein bottle, and a Möbius sheet it is a torus; etc.) Not every closed orientable surface is the double of something. Usually, an antiholomorphic Z_2 transformation of the type $z\rightarrow\bar{z}$ sends one surface into some other surface: Acting on the space M_p is its own antiholomorphic Z_2 isometry. The space of moduli of doubles M_p^D is a set of fixed points of this isometry in M_p.

The action of an antiholomorphic isometry can be specified in a particularly simple way not on M_p itself but on a Siegel space σ_p: a set of matrices of periods[1] into which M_p fits in a holomorphic way. More precisely, $M_p\subset\sigma_p/\mathrm{Sp}(p,Z)$. The isometry acts on σ_p in accordance with the rule $T\rightarrow -\bar{T}$. Here M_p^D is the intersection of M_p with the set of fixed points of this mapping, and $T=[S(-\bar{T})+R]/[-P(-\bar{T})+Q]$, where $\left(\begin{smallmatrix} S & R \\ -P & Q\end{smallmatrix}\right)\in\mathrm{Sp}(p,Z)$ is a simplectic matrix whose square is unity. It follows that $S=-\tilde{Q}$. This equation on the matrix of periods is equivalent to the simpler equation

$$(PT+Q)(P\bar{T}+Q)=I\ , \tag{2}$$

and various choices of the $p \times p$ matrices P and Q [restricted by the condition of a simplectic nature, $\left(\begin{smallmatrix} -Q & R \\ -P & Q \end{smallmatrix}\right) \in \mathrm{Sp}(p,Z)$] determine various components of the space of moduli of the doubles[1)] M_p^D.

These components are the spaces of moduli $M_{h,c,m}$. If a Z_2-isometry of a double does not have fixed points, then a nonorientable closed surface ($m = 0$) is found from it after a factorization. If the Z_2-isometry has fixed points, then they form the boundary of a surface with an edge. If fixed lines separate a surface into two unconnected components, we find an orientable surface with an edge ($c = 0$). If, after fixed lines are discarded, the double remains connected, then the factorization results in the appearance of a nonorientable surface with an edge: $c,m \neq 0$.

The unknown measure of integration on $M_{h,c,m}$, which determines the string partition functions, is expressed in terms of $d\mu_{\text{bos}}^{(p)}$, which appears in (1) for closed unoriented strings ($p = 2h + c + m - 1$):

$$d\nu_{h,\,c,\,m} = [f_{\pm}(P,\,Q)/\det(PT + Q \pm I)]^{13\,(14)}\,d\mu_{bos}^{(p)}\,. \qquad (3)$$

The factors $f_{\pm}\,(P,Q)/\det(PT + Q \pm I)$ can be described geometrically in the following way. We recall that the quantity $\det \mathrm{Im} T$ in (1) can be characterized as the volume of a Jacobian with respect to the measure of $\omega\bar{\omega}$, where ω is a p-form—a product of basis holomorphic 1-differentials normalized to the A-periods. On the Jacobian of a surface, which is a double, there is also a natural action of antiholomorphic Z_2 symmetry. We consider two real p-dimensional subtori in a Jacobian, one consisting of invariant points, and the other of points that change sign as a result of this action. The quantities $\det(PT + Q \pm I)/f_{\pm}$ are then given by integrals of ω over these subtori. (For type I, for example, the numbers $f_{\pm}$ are the largest common divisors of the numbers P and $Q \pm 1$.) We thus have $\det(PT + Q + I)\det(P\bar{T} + Q - I)/f_+ f_- \sim \det \mathrm{Im} T$, and, as we see, the measure $d\nu_{h,c,m}$ is very similar to the "square root" of $d\nu_p$. The factors $\det(PT + Q \pm I)$ complement the Mumford measure $d\mu_{\text{bos}}^{(p)}$ to a real expression on $M_{h,c,m}$. They also provide modular covariance: Under transformations from the $\mathrm{Sp}(p,Z)$ group, expression (3) goes into itself, but with altered matrices P and Q. Modular transformations which do not alter Eq. (2) replace the plus sign by the minus sign in (3). The space M_p^D is shown for $p = 1$ in Fig. 1 (A. Tseĭtlin has informed us that the type I case has already been analyzed in the literature[3]). The proofs of these assertions, including a derivation of expression (3) from a path integral, will be published in a more detailed paper.

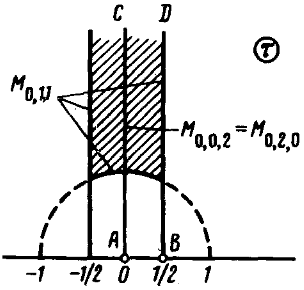

FIG. 1. The space M_{I} (hatched) and the nesting in it of the modulus spaces of rings ($M_{0,0,2}$), Klein bottles ($M_{0,2,0}$), and Möbius sheets ($M_{0,1,1}$).

The description of partition functions for open and/or oriented strings in terms of the spaces M_p given above determines, among other things, a natural relative asymptotic form of the expressions for the various string diagrams near the boundaries of modulus spaces. (In other words, this description ensures a matched regularization of these expressions.) For $p = \mathrm{I}$, for example (Fig. 1), the expressions for $d\nu_{0,0,2}$ (a ring) and $d\nu_{0,1,1}$ (Möbius sheet) should be the same in the unitary limit: at points A and B (here a Riemann surface degenerates into an infinitely thin strip, and an expansion in string excitations "works"). In the ultraviolet limit—at points C and D—the relative normalization of the two components then differs by a factor of 2^{14}. [This result is a simple consequence of the fact that under modular transformations which send the point A into C, $\tau \to -(1/\tau)$, and which send the point B into D, $\tau \to (\tau - 1)/(2\tau - 1)$, the form of weight 14 acquires factors τ^{14} and $(2\tau - 1)^{14} = 2^{14}(\tau - 1/2)^{14}$, respectively.] When we go to superstring amplitudes, the factor 2^{14} becomes $2^5 = 32$, which leads to a cancellation of the divergences and anomalies for open orientated strings with gauge group $SO(32)$.

It would also be interesting to find a description of the modulus spaces of surfaces with points of the type mentioned above, which correspond to the amplitudes for the scattering of string excitations, in terms of spaces M_p with higher values of p. These topics will be discussed in more detail in a future paper.

[1)] The following, more elegant description of M_p^D apparently holds here: geodetic [in the sense of a metric induced by the invariant metric of $\mathrm{Tr}\, dT(\mathrm{Im}T)^{-1}\, \overline{dT}\,(\mathrm{Im}T)^{-1}$ on σ_p] subspaces of half the dimensionality in M_p, which intersect at singular points of M_p.

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[3] C. P. Burgess and T. R. Morris, I.A.S. Preprints, 1986.

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