Polarization domains in nonlinear optics

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Stable (ground) polarization states have been found in the case of the propagation of electromagnetic waves in opposite directions in a mirror-symmetry medium with a cubic nonlinearity along a fourfold axis if the optical Kerr self-effect is ignored. The evolution of an arbitrary initial polarization gives rise to a "domain structure." Explicit solutions describing domain walls are found.

We consider quasimonochromatic electromagnetic waves with characteristic frequencies $\omega^{\pm}$ and complex envelopes $E^{\pm}(x,t)$ which are propagating in opposite directions at group velocities $v_{\pm}$ in a medium with a cubic nonlinearity which is either anisotropic (along a fourfold axis) or isotropic. The mutual changes in the polarizations of these waves constitute the strongest nonlinear effect. The static problem of the spatial evolution of polarizations was discussed in Refs. 1–3. In the present letter we show that when the Kerr self-effect of the waves is ignored, the problem can be solved exactly in a general time-dependent formulation. The problem reduces in this case to an integrable system of an anisotropic chiral field on group O(3). This circumstance means, in particular, that the asymptotic behavior of an arbitrary initial state of the polarizations will have a "domain" structure, in which the domains are regions with different stable static states which realize minima of the Hamiltonian. There will also be domain walls, whose size and velocity will depend on the amplitudes of the incident waves and the nonlinear interaction constants. These walls are regions of a switching of polarizations. It is clear that this domain structure will be preserved if we incorporate a damping of waves or other nonlinearities which do not disrupt the degeneracy of a discrete set of minima of the Hamiltonian.

Assuming $E^{\pm}(x,t) = E_y^{\pm} j + E_z^{\pm} k$, we write the general phenomenological interaction Hamiltonian as $H_{\rm int} = H^{+} + H^{-} + \hat{H}$, where

$$H^{\pm} = \frac{1}{2}\int dx\,\{\beta_1^{\pm}|E^{\pm}|^4 + \beta_2^{\pm}(|E_y^{\pm}|^4 + |E_z^{\pm}|^4) + \beta_3^{\pm}[(E_y^{\pm})^2(E_z^{\pm}{}^*)^2 + \text{c.c.}]\},$$

$$\hat{H} = \int dx\,[\gamma_1|E^{+}|^2|E^{-}|^2 + \gamma_2(|E_y^{+}|^2|E_z^{-}|^2 + |E_z^{+}|^2|E_y^{-}|^2)$$

$$+\gamma_3(E_y^{+}E_z^{+}{}^*E_y^{-}E_z^{-}{}^* + \text{c.c.}) + \gamma_4(E_y^{+}E_z^{+}{}^*E_y^{-}{}^*E_z^{-} + \text{c.c.})].$$

After we transform to real variables which are quadratic in the fields,

$$S_0^{\pm} = |E^{\pm}|^2,\quad S^{\pm} = (E_y^{\pm}{}^*E_z^{\pm} + \text{c.c.},\, iE_y^{\pm}{}^*E_z^{\pm} + \text{c.c.},\, |E_y^{\pm}|^2 - |E_z^{\pm}|^2), \tag{1}$$

the system of equations of motion, $\partial_t E^{\pm} \pm v_{\pm}\partial_x E^{\pm} = i\delta H_{\rm int}/\delta E^{\pm}{}^*$, takes the form $\partial_\xi S_0^+ = 0, \partial_\eta S_0^- = 0$,

$$\partial_\xi S^+ = S^+ \times (J_+ S^+ + J S^-), \quad -\partial_\eta S^- = S^- \times (J_- S^- + J S^+), \tag{2}$$

where $\partial_\xi = \partial_t + v_+ \partial_x$, $\partial_\eta = -\partial_t + v_- \partial_x$, $J_\pm = \mathrm{diag}(-\beta_3^\pm, \beta_3^\pm, -\beta_2^\pm)$, and $J = \mathrm{diag}(-\gamma_3 - \gamma_4, \gamma_3 - \gamma_4, \gamma_2)$. The interactions with the constants $\beta_1^\pm$ and γ_1 are inconsequential: they simply introduce a nonlinear correction to the isotropic part of the refractive index. They are automatically eliminated in terms of variables (1). In the case of an isotropic medium we would have

$$\beta_2^\pm = \beta_3^\pm, \quad \gamma_2 = -\gamma_3 - \gamma_4. \tag{3}$$

At this point we restrict the discussion to situations in which we can ignore the optical Kerr self-effect of the waves E^+ and E^- in comparison with their interaction; i.e., we assume $|E^+|^2 |E^-|^2 \|J\| \gg |E^\pm|^4 \|J_\pm\|$. Such a situation arises when the frequency difference $\omega^+ - \omega^-$ is close to a natural frequency of the medium.[4] In this case, system (2) simplifies, becoming the well-known model of an anisotropic chiral field on group O(3) (a model which is integrable by the method of the inverse problem)[5,6]:

$$\partial_\xi S^+ = S^+ \times J S^-, \quad -\partial_\eta S^- = S^- \times J S^+, \quad J = \mathrm{diag}(J_1, J_2, J_3). \tag{4}$$

Static states correspond to constant vectors S^+ and S^-, which are directed along one of the principal axes of the tensor J. We first assume $J_a^2 \neq J_b^2$ with $a \neq b$. Analysis of the dispersion relation for small oscillations near each of these equilibrium states shows that the only stable state is that in which both of the vectors S^+ and S^- are oriented along principal axis e_a of the tensor J_a with the smallest square eigenvalue ($J_a^2 < J_b^2, J_c^2$). These vectors are parallel if $J_b J_c > 0$ or antiparallel if the product $J_b J_c$ is negative. Parallel vectors $S^\pm$ correspond to identical polarizations of the waves $E^\pm$, while antiparallel vectors correspond to orthogonal polarizatins; i.e., $(E^{+*}, E^-) = 0$. Table I shows pairs of stable polarization states of the field E^+, E^- for various relations between the interaction constants. The waves tend to arrive at these states, which are the most favorable states from the energy standpoint; i.e., the waves

TABLE I.

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<tr><th rowspan="2"></th><th colspan="2">$J_1^2 < J_2^2, J_3^2$</th><th colspan="2">$J_2^2 < J_1^2, J_3^2$</th><th colspan="2">$J_3^2 < J_1^2, J_2^2$</th></tr>
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The plus and minus signs specify whether the circular polarization is positive or negative; the bars show the orientation of the linear polarization in the (y,z) plane; the polarizations are given in the order E^+, E^-.

tend to undergo a self-polarization. The situation here is completely analogous to the theory of ferromagnetism, in which spins tend to become aligned along the most favorable anisotropy axis. Also as in ferromagnets, there are domain walls here, which describe a transition from one stable state to another. In the case $J_1^2 > J_2^2 > J_3^2, J_1 J_2 > 0$, for example, this wall is described by

$$S^+ = (a \sin\varphi/\cosh X,\ a\cos\varphi/\cosh X,\ a\tanh X),$$

$$S^- = (b\sin\psi/\cosh X,\ b\cos\psi/\cosh X,\ b\tanh X), \qquad (5)$$

$$X = \frac{p b v_- + q a v_+}{v_+ v_-}\left[x - \frac{(qa - pb)\, v_+ v_-}{(v_+ qa + v_- pb)}\, t - x_0\right],$$

where x_0, a, and b are arbitrary constants; and the constant parameters p, q, φ, and ψ are related by the three relations

$$p + q = (J_1 + J_2)\sin(\varphi - \psi), \quad p - q = (J_2 - J_1)\sin(\varphi + \psi), \qquad (6)$$

$$J_3 = J_2 \cos\varphi\cos\psi + J_1 \sin\varphi\sin\psi .$$

Consequently, for given wave intensities and a given value of x_0, there is a single-parameter family of solutions in the form of a domain wall.

System (4) and the Landau-Lifshitz model describing a biaxial ferromagnet are integrated with the help of the same linear spectral problem (cf. Refs. 5 and 7). Accordingly, all the results found for magnetic materials can be transferred without difficulty to our case. For example, we can write exact N-soliton and finite-band periodic solutions. We can study the interaction of domain walls with packets of polarization waves (analogs of spin waves), etc. The stability of the polarization domains is of a topological nature, and the radiation power levels and sample dimensions required for observing them are accessible experimentally.

Let us examine in more detail the structure of domains in isotropic media. In this case there is a degeneracy of the eigenvalues of the tensor J in (3). If $J_1^2 = J_3^2 > J_2^2$, the ground states of Hamiltonian $\hat{H}$ are identical positive or negative circular polarizations of the fields $E^\pm$. A domain wall couples states with opposite circular polarizations (Fig. 1). In the interior of a wall, the polarizations of both waves are elliptical,

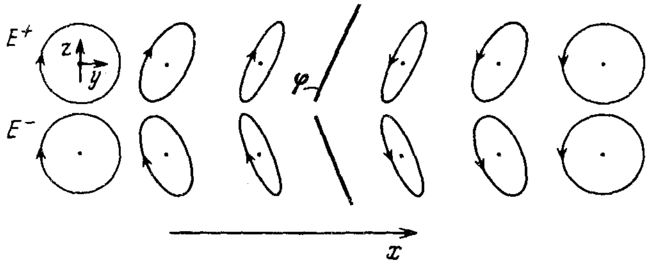

FIG. 1. Polarization ellipses of the waves E^+ and E^- in a domain wall in an isotropic medium.

and at the very center of the wall they are linear. The ratios of the minor ($b_2^{\pm}$) and major ($b_1^{\pm}$) axes of the polarization ellipse are the same for the two waves and are given by

$$\frac{b_2^{\pm}}{b_1^{\pm}} = \tanh\left[\sqrt{\gamma_3\gamma_4}\,(|E^+|^2+|E^-|^2)\left(x-\frac{|E^+|^2-|E^-|^2}{|E^+|^2+|E^-|^2}\,t-x_0\right)\right]. \qquad (7)$$

(For simplicity, we have set $v_{\pm} = 1$ here.) Over the entire thickness of a domain wall the polarization ellipses of the two waves retain the same orientation. Their major axes are separated from each other by an angle $\varphi = \pm(1/2)\arctan[2\sqrt{\gamma_3\gamma_4}/|\gamma_3-\gamma_4|]$. The orientation of the polarization ellipse of one of the fields is arbitrary, as it should be in an isotropic medium. The velocity of a domain wall in this case is determined completely by the intensities of waves E^+, E^-; it vanishes if these intensities are equal.

If the opposite inequality holds, $J_1^2 = J_3^2 < J_2^2$, the Hamiltonian is infinitely degenerate, the fields $E^{\pm}$ are identically linearly polarized in the ground state, and the orientation of the polarization in the (y,z) plane is arbitrary. In the theory of ferromagnetism, the first case corresponds to an easy-axis anisotropy, while the second corresponds to an easy-plane anisotropy. The case $J_1 = J_2 = J_3$ (which leads to the model of a principal chiral field[8]) is even more degenerate: An arbitrary identical polarization of waves $E^{\pm}$ is of neutral stability. This case occurs in a plasma if $\omega^{\pm} \gg \omega_p$.

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