1/*n* expansion and wave functions

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The $1/n$ expansion is used to calculate wave functions. Analytic expressions are derived. They are valid for an arbitrary smooth potential $V(r)$, in particular, for the asymptotic coefficients in the limits $r\to 0$ and $r\to\infty$. Comparison with numerical calculations shows that the wave function is calculated very accurately at the origin.

1. The $1/n$ expansion is used in various fields of theoretical physics, including quantum mechanics, statistical physics, and field theory.[1–4] It has been shown[5] that this expansion can be used to find the energies and widths of resonance states. Use of the $1/n$ expansion has also proved successful in the theory of the strong-field Stark effect.[6]

In those papers, a $1/n$ expansion was carried out for the energies of levels:

$$\epsilon \equiv 2n^2 E_{nl} = \epsilon^{(0)} + \frac{\epsilon^{(1)}}{n} + \frac{\epsilon^{(2)}}{n^2} + \ldots \tag{1}$$

The first term, $\epsilon^{(0)}$, corresponds to the energy of a clasical particle which is at rest at the point at which the effective potential $U(r) = V(r) + l(l+1)/2r^2$, reaches its minimum; the next coefficients $\epsilon^{(k)}$ are calculated with the help of recurrence relations.

It is natural to use the $1/n$ expansion to calculate wave functions and the like. Some results achieved in this direction are reported in Refs. 7 and 8, but here we will take a different approach. We will not attempt to carry out highly accurate calculations, so we will find analytic expressions which can be used to calculate the wave functions $\psi_{nl}(r)$ within a reasonable error ($\lesssim 1\%$), including the wave functions near singular points ($r\to 0$, $r\to\infty$). Problems of this sort arise frequently in physics.[1] In a calculation of $\psi_{nl}(0)$ through a numerical solution of the Schrödinger equation, computational difficulties arise for states with large l, while the $1/n$ expansion converges better, the larger l.

2. Wave functions and $\psi(0)$. We consider the screened Coulomb potential

$$V(r) = -r^{-1} f(\mu r), \qquad \hbar = m = e = 1. \tag{2}$$

Assuming $x = \mu r$, $\nu = n^2\mu$, $\epsilon = 2n^2 E_{nl}$, $\rho = (l+1/2)/n$, and $\psi_{lm}(r) = r^{-1}\chi_l(r) Y_{lm}(r/r)$, we find from the Schrödinger equation

$$\frac{d^2\chi_l}{dx^2} - n^2 Q^2(x)\,\chi_l = 0, \quad Q(x) = \left[\frac{\rho^2}{x^2} - \frac{2f(x)}{\nu x} - \frac{\epsilon}{\nu^2}\right]^{1/2} \tag{3}$$

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for $n \to \infty$ and $k = n - l - 1 \ll n$ that the particle is near the classical equilibrium point $x = x_0 = \mu r_0$, which is determined from the equation[5] $xf - x^2 f' = \nu$. We accordingly set $x = x_0(1 + \xi n^{-1/2})$, in (3); ξ remains on the order of unity as $n \to \infty$. In the region $|\xi| \ll n^{1/2}$, $\chi_l(x)$ is the same as the wave function of level k of a harmonic oscillator with a frequency ω given by

$$\omega = \left[\frac{h(x_0)}{g(x_0)} \right]^{1/2}, \quad g = f - xf', \quad h = f - xf' - x^2 f''. \tag{4}$$

The WKB method can be used to continue this function into the tunneling region. In particular, we can write

$$\chi_{nl}(r) = c_{nl} \mu^{l+3/2} r^{l+1} + \ldots, \quad r \to 0, \tag{5}$$

where $\int_0^\infty \chi_{nl}^2(r) dr = 1$, and c_{nl} is a dimensionless coefficient. We restrict the discussion here to the case $k = 0$ ($n = l + 1$; states without nodes):

$$c_{n,n-1} = \left(\frac{n\omega^3}{\pi x_0^2} \right)^{1/4} \exp\{ -(nJ_0 + J_1) \}, \tag{6}$$

where

$$J_0 = \ln x_0 + \int_0^{x_0} dx \, [Q_0(x) - x^{-1}],$$
$$J_1 = \frac{1}{2} \int_0^{x_0} dx \left\{ \frac{x_0}{x(x_0 - x)} - [x^{-2} + (\omega - 1) x_0^{-2}] \, Q_0^{-1}(x) \right\} \tag{6'}$$

ω is defined in (4), $Q_0^2(x) = x^{-2} - [2f(x)/\nu x] - (\epsilon^{(0)}/\nu^2)$, and $\epsilon^{(0)} = (x^2 f'^2 - f^2)|_{x = x_0}$. Let us consider some examples.

3. For power-law potentials

$$V(r) = r^N/N, \quad \mu = 1, \tag{7}$$

expression (6) is very accurate even for the ground state. Denoting by ρ_l the ratio of approximate coefficient (6) to the exact coefficient $c_{l+1,l}$, we have $\rho_0 = 1.0209$ for $N = -1$ (a Coulomb potential), $\rho_0 = 0.9952$ for $N = 1$, $\rho_0 = 0.9803$ for $N = 2$ (a harmonic oscillator), etc. In the limit $l \to \infty$ we have $\rho_l \to 1$. For example, we have $\rho_l = 1 + (48l)^{-1} + \cdots$ for $N = -1$.

A funnel potential

$$V(r) = -\frac{\kappa}{r} + \frac{r}{a^2} \tag{8}$$

is frequently used to describe quarkonium and multiquark systems. In this case $\mu = (\kappa a^2)^{-1/2}$, $f(x) = 1 - x^2$, $\omega = [(1 + 3x_0^2)/(1 + x_0^2)]^{1/2}$, x_0 is determined from the equation $x^3 + x = n^2 \mu$, and the integrals J_0 and J_1 can easily be evaluated analyti-

TABLE I. Accuracy of the $1/n$ expansion in the limit $r \to 0$ (a funnel potential).

l	λ	$c_{l+1,l}$	C_l	
			Acc. expression (6)	Numerical calculation [11]
1	0.43768	2.197 (−1)	5.940 (−2)	5.955 (−2)
1	2.74036	5.290	1.430	1.43317
3	2.22329	6.943 (−1)	1.940 (−2)	1.943 (−2)
4	2.87926	6.152 (−1)	1.559 (−3)	1.561 (−3)
6	3.56631	1.944 (−1)	1.675 (−4)	1.677 (−4)

The order of magnitude of the number is given in parentheses: 2.197 (− 1) = 0.2197; etc.

cally. The results found from (6) are compared with results of numerical calculations[11] in Table I. Given here in addition to l are the values of the Coulomb parameter[10] $\lambda = \kappa(2m_q a)^{2/3}$. The coefficients $c_{l+1,l}$ are the dimensionless coefficients calculated from (6); the c_l are of dimensionality $(\mathrm{GeV})^{l+3/2}$ and are determined by the normalization adopted in Ref. 11. There is a strikingly good agreement between approximation (6) and the exact coefficients even at small values of l.

Using (6), we can easily analyze the behavior of the wave function at the origin for various versions of the potential $V(r)$. For example, the blocking potential in (8) can be turned on at $r > r_1$ (this situation corresponds to the case in which a string between quarks forms only at distances $r \geqslant r_1$). It follows from (6) that $\psi(0)$ increases here {we have $x_0 \approx \nu^{1/3} \gg 1$ at $n \gg 1$, and the coefficient $c_{n,n-1}$ increases by a factor of $1 + (4n/3)[1 + (1/4x_0^2)]r_1/r_0$}.

4. Corresponding results can be derived for the asymptotic coefficient $\chi_l(r)$ in the limit $r \to \infty$. We restrict the discussion to states with a zero energy [the time at which the first l level appears in the potential $V(r)$]. In this case we have

$$\chi_l(r) \underset{(r\to\infty)}{=} A_l r^{-l} + \ldots, \qquad r_l = -2^{1-2l}[(2l)!/l!A_l]^2, \tag{9}$$

where χ_l is a normalized ($l \neq 0$) wave function, and the effective radius r_l is an important parameter in the theory of low-energy scattering (Ref. 14, for example). From the $1/n$ expansion we find an expression like (6) for A_l, and for the effective radius we find

$$r_l = -\frac{4(2l)!}{2l+1}\left(\frac{x_0^2}{\omega}\right)^{3/2} \exp\{-[(2l+1)I_0 + I_1]\}, \tag{10}$$

where $l \geqslant 1$ and

$$I_0 = \ln x_0 + \int_{x_0}^{\infty} \frac{dx}{x} \left\{ 1 - \left[1 - \frac{xf(x)}{x_0 f(x_0)} \right]^{1/2} \right\}.$$

The expression for I_1 is more lengthy, and to save space we will not reproduce it here. For the Yukawa potential $f(x) = \exp(-x)$ we have $x_0 = 1$, $\omega = 2^{-1/2}$ (at $E = 0$), $I_0 = 0.730$, and $I_1 = -0.203$. Comparison of (10) with the numerical calculations of Ref. 13 shows that this expression is not accurate[2)] for p and d waves, but it becomes more accurate as l is increased [at $l = 4$, for example, we have $r_l = -67.2\mu^7$, according to Ref. 13, and $r_l = -52\mu^7$, according to (10)]. Expression (10) can thus be used in the case $l \gg 1$.

5. Expressions (6) and (10) can be generalized to states with nodes ($k = 1, 2, \ldots \ll n$). We would also like to use the $1/n$ expansion to find the asymptotic behavior of $\chi_l(r)$ as $r \to \infty$ for bound states ($E < 0$). We hope to return to these questions in the future.

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[1)]The values of $\psi^2(0)$ are required for calculating the annihilation widths of quasinuclear systems (of the $\bar{p}p$ type[9]), the decay widths of charmonium[10] and of four-quark bound states,[11] etc. The wave function in the limit $r \to \infty$ is used, for example, in calculating an effective radius.[12,13]

[2)]Apparently because expression (10), in contrast with (6), refers to the case $E = 0$. For bound states, the accuracy of the corresponding expression should improve with increasing binding energy.

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