Quantization of gauge theories with nilpotent generators

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The problem of the covariant first quantization of a superparticle and a superstring is shown to be related to the nilpotency of the gauge generators of first-generation ghosts. A correct quantization of such theories is carried out. This quantization includes, in addition to the usual latitude regarding the choice of gauge, some latitude regarding the choice of the number of generations of ghosts.

As we know, the Green-Schwarz covariant action of a superstring[1] can be quantized in only *one* light-cone gauge, with a breaking of the Lorentz invariance which is global over the world sheet and a breaking of supersymmetry. In this letter we find the reasons for this situation, which is quite unusual for a gauge theory, and we find a method for quantizing theories of the superstring type in an arbitrary gauge.

The classical action $\mathscr{S}(\varphi)$ of a gauge theory is invariant under gauge transformations $\delta\varphi^i = R^i_{\mu_1}\xi^{\mu_1}$. If the vectors $R^i_{\mu_1}$ are linearly independent at a stationary point $\mathscr{S}_{,i}[\varphi_0] = 0$, the theory is described by a single generation of ghosts. If the generators $R^i_{\mu_1}$ have a null vector

$$R^i_{\mu_1} Z^{\mu_1}{}_{1\,\mu_2}\Big|_{\varphi_0} = 0, \tag{1}$$

the ghosts of the first generation become gauge fields, and ghosts of a second generation arise. If Z_1 in turn has a null vector, $Z_{1\,\mu_2}^{\mu_1} Z_{2\,\mu_3}^{\mu_2} = 0$, the ghosts of the second generation become gauge ghosts, and ghosts of a third generation arise (and so forth). In the general case of an unclosed algebra we know how to quantize a theory with a *finite* number of generations, in which the ghosts of the last generation are no longer gauge fields.[2]

However, neither the action of a superparticle[3] nor the action of a superstring[1,4] falls in this category. In these actions, the initial generators have a null vector $Z^{\mu_1}_{\mu_2}$, which is nilpotent at a stationary point:

$$Z^2\,\Big|_{\varphi_0} = 0. \tag{2}$$

This assertion means that the ghosts of any generation are gauge fields, and we do not have a quantization recipe for the ghosts of a theory with an infinite number of ghosts. In order to devise a procedure for quantizing such theories, we make use of some specific properties of the generators R and Z which are characteristic of superparticle and superstring theories[1,3,4]:

A) The null vector $Z^{\mu_1}_{\mu_2}$ is nontrivial for only part of [1)] μ_1. The corresponding part

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of the gauge generators of the gauge symmetry, R^i_α, does not contain differential operators.

B) The nilpotent operator Z does not contain differential operators.

As a starting point we use the construction which Batalin and Vilkovisky[2] developed for a theory with a finite number of generations. A functional integral for the gauge action $\mathscr{S}(\varphi)$ is constructed in the following way (to an accuracy within the local measure of integration):

$$\mathcal{I} = \int \exp\frac{i}{\hbar} S(z)\, \delta(\chi_A(z))\, J^{1/2} \prod_{\mathcal{A}} dz^{\mathcal{A}} \ , \tag{3}$$

where $z^{\mathcal{A}} = (\Phi^A, \Phi^*_A)$, $A = 1,...,N$; and $\mathcal{A} = 1,...,2N$. The fields Φ^A include a minimal set of classical fields φ and ghosts of m generations, C_s, $s = 1,...,m$; and also a set of s fields for each ghost of the same statistics: antighosts $\overline{C}_s$ and $(s-1)$ extraghosts $C_s^{(1)},...,C_s^{(s-1)}$. In addition, for each $\overline{C}_s, C_s^{(1)},...$ we add fields $\pi_s, \pi_s^{(1)},...,\pi_s^{(s-1)}$ of the opposite statistics: Nakanishi-Lathrop fields. Furthermore, with each field Φ^A we associate an antifield Φ^*_A. The quantities $S(z)$ and $\chi_A(z)$ in (3) satisfy the equations

$$(S, S) = 0, \tag{4}$$

$$(\chi_A, \chi_B) = 0, \tag{5}$$

where the antibrackets are defined by

$$(X, Y) = \frac{\partial_r X}{\partial z^{\mathcal{A}}} \omega^{\mathcal{A}\mathcal{B}} \frac{\partial_e Y}{\partial z^{\mathcal{B}}} = \frac{\partial_r X}{\partial \Phi^A} \frac{\partial_e Y}{\partial \Phi^*_A} - \frac{\partial_r X}{\partial \Phi^*_A} \frac{\partial_e Y}{\partial \Phi^A} \ . \tag{6}$$

We can choose S and χ_A to be of the form

$$S(\Phi, \Phi^*) = \mathscr{S}(\varphi) + \sum_{n=1} \Phi^*_{A_n} \ldots \Phi^*_{A_1} S^{A_1 \ldots A_n}(\Phi), \tag{7}$$

$$\chi_A = \Phi^*_A - \frac{\partial \Psi}{\partial \Phi} \ . \tag{8}$$

Equation (4) then determines the coefficients $S^{A_1 \ldots A_n}$ in (8), while (5) is satisfied automatically. The determinant of the canonical transformation, $J^{1/2}$, is unity.

It follows from (3) that the maximum latitude in the quantization of a gauge theory with m generations of ghosts consists of (first) the possibility of a canonical replacement of the variables Φ and Φ^* and (second) the choice of the function Ψ in (8), which determines the choice of the gauge condition.

We can show that superparticle and superstring theories[1,3,4] can be described correctly by a functional integral which is of the type in (3) but in which the number of fields $z^{\mathcal{A}}$—i.e., the number of generations of ghosts—is arbitrary to the same extent that the choice of the gauge function is arbitrary.

This new procedure for quantizing theories (1), (2), A), B) consists of constructing functional integral (3) as outlined above and imposing an arbitrary *algebraic* condition of the following type on the ghosts of the m-th generation[2)]:

$$\sigma^a_{\ \alpha} C^\alpha_{\ m} = 0, \qquad C_m^{\ \alpha} = \tilde{\sigma}^\alpha_{\ a} C_m^{\ a}, \qquad \sigma^a_{\ \alpha} \tilde{\sigma}^{\alpha b} = 0 . \tag{9}$$

Conditions (9) mean that the ghosts of the m-th generation are no longer gauge fields; i.e., the theory has been reduced to a theory with an *arbitrary* but finite number of generations.[2] We can show that (1) the theory does not depend on the number of ghost generations or the cutoff method, i.e., on the values of σ^a_α, and (2) the theory is equivalent to a theory which is quantized in a unitary gauge.

A unitary gauge is found if condition (9) is imposed on the ghosts of the first generation. In this case a quantization in accordance with the rules given above means that terms

$$\chi^a(\varphi^i)\pi_a + C_a \frac{\partial \chi^a}{\partial \varphi^i} R^i_{\ \alpha} \tilde{\sigma}^\alpha_{\ b} C^b , \tag{10}$$

appear in the quantum action. If the gauge function $\partial\chi^a/\partial\varphi^i$ does not contain differential operators [and R^i_α does not contain them by virtue of condition A)], then it is obvious that the introduction of ghosts is superfluous, since an integration over them leads to simply a change in the local measure of integration, $\sim\delta^n(0)$ ($n = 2$ for a string). In this case we find a unitary gauge. The dependence on $\tilde{\sigma}^\alpha_b$ also drops out. If, on the other hand, $\partial\chi^a/\partial\varphi^i$ contains differential operators, then the ghost of even the first generation in (10) "come to life," and kinetic terms are found.[3)]

We can show that this new quantization of a theory with m generations is equivalent to a quantization with $m-1$ generations of ghosts. Conditions A) and B) and the special choice of the function Ψ_m in (8) make it possible to carry out a functional integration over all the fields of the m-th generation. In this case, an integration over $\overline{C}_m$, C_m yields a result proportional to $\delta^n(0)$. An integration over the m Nakanishi-Lathrop fields $\pi_m, \pi_m^{(1)}, \ldots, \pi_m^{(m-1)}$ leads to limitations on the m fields of ghosts, antighosts, and extraghosts of the $(m-1)$-st generation, i.e., on $C_{m-1}, \overline{C}_{m-1}, C^{(1)}_{m-1}, \ldots, C^{(m-2)}_{m-1}$. Finally, an integration over the $(m-1)$ extraghosts of the m-th generation, $C^{(1)}_m, \ldots, C^{(m-1)}_m$, gives us the limitation which we need on the $(m-1)$ Nakanishi-Lathrop fields of the $(m-1)$-st generation. At an accuracy within the local measure of integration, we thus find a correct description of the theory as a theory with $m-1$ generations, in which all of the gauge conditions except $\tilde{\sigma}^a_\alpha C^\alpha_{m-1}$ can easily be replaced in the standard way by changing the function Ψ_{m-1}; the dependence on the method of "cutting off the tail," i.e., on σ^a_α, appears only in the terms with $\delta^n(0)$, as we have already mentioned.

Let us illustrate this mechanism for the transition $m = 3 \to m = 2$; i.e., let us consider (9) for $m = 3$. We choose the gauge function to be of the form $\Psi_3 = \Psi_1 + \overline{C}_{3a}\omega^a_\alpha C^\alpha_1 + \overline{C}_{2\alpha}\eta^\alpha_a C^{(1)a}_3 + \overline{C}^{(2)}_{3a}\zeta^a_{\ \alpha} C^{(1)\alpha}_2$; Ψ_2 contains ghosts of only the first and second generations; and ω, η, ζ do not contain differential operators. An integration over π_{3a}, $\pi^{(1)a}_3$, $\pi^{(2)}_{3a}$ yields $\omega^a_\alpha C^\alpha_2 = \overline{C}_{2\alpha}\eta^\alpha_a = \zeta^a_{\ \alpha} C^{(1)\alpha}_2 = 0$, while an integration over $C^{(1)a}_3$, $C^{(2)a}_3$ correspondingly yields $\zeta^a_{\ \alpha}\pi^{(1)\alpha}_2 = \pi_{2\alpha}\eta^\alpha_a = 0$; these are the relations which we need for the ghosts of the second generation in the case in which the second generation is the last. The dependence on $C_3^a, \overline{C}_{3a}$ is of the type $\overline{C}_{3a}\omega_{\underline{\alpha}}^a Z^{\alpha\beta}\tilde{\sigma}_b^\beta C_3^b$, and by virtue of condition B) and our choice of ω, $\tilde{\sigma}$ the integration over $\overline{C}_3$, C_3 contributes to only the local measure.

In summary, we have in principle solved the problem of quantizing theories with nilpotent generators with the properties specified above. The specific implementation of this quantization method in superparticle and superstring theories will be the subject of a separate paper.

It is a privilege to express my deep gratitude to I. A. Batalin for many discussions of the fundamentals of quantization.

[1)]We will denote it below as α, bearing in mind the local supersymmetry $Z^{\alpha\beta} = \gamma^k p^k$. The ghosts of the reparameterization are not gauge ghosts.
[2)]In the case of a superparticle and a superstring we would have $\alpha = 1,...,16$; $\alpha = 1,...,8$.
[3)]In the theories of Refs. 1, 3, and 4, χ^a can be $(\gamma^0 + \gamma^9)\theta$ or $(\gamma^0 + \gamma^9)(\partial_\tau + \partial_\sigma)\theta$, respectively; in the latter case, global supersymmetry is conserved.

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