

# Universality of helium

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At temperatures  $T > 3$  K the thermodynamic functions of liquid helium depend universally on the reduced temperature  $\tau = T/\bar{K}^0$ . The dependence of the energy of zero-point oscillations  $\bar{K}^0$  on the density of the liquid and the atomic mass is determined from the experimental data at room temperature.

It was shown in Refs. 1–3 that in the case of liquid  $\text{He}^3$  and  $\text{He}^4$  there is a broad range of quantum phenomena which are insensitive to the statistical base of the particles. The condition under which condensed matter is a quantum liquid and the condition under which it is a degenerate liquid contradict each other: The quantum level of a medium increases as its degeneracy decreases, because even at  $T = 0$  the wavelength of an atom of a dense liquid is shorter than the average interatomic distance  $a$ . The characteristic quantum scale of a liquid is the energy of its zero-point vibrations  $\bar{K}^0$ , i.e., the average kinetic energy of the atom  $\bar{K}$  at  $T = 0$

$$\bar{K}(n, T) = \bar{K}^0 K^*(n, \tau), \quad \tau = T/\bar{K}^0, \quad \bar{K}^0 = \bar{K}(n, 0). \quad (1)$$

Since the principal dependence of  $\bar{K}$  on the density  $n$  is given as a scaling factor of  $\bar{K}^0$ , the reduced kinetic energy  $K^*$  depends essentially on the reduced temperature  $\tau$  and

depends only very slightly on  $n$ . In other words,  $K^*$  is a universal single-parameter function  $\tau$  which is identical for all quantum liquids within small exchange effects. This circumstance renders helium thermodynamics and kinetics universally applicable over extremely broad temperature range  $3 < T < 10^3$  K and pressure range  $P < 10^4$  bar. The experimental data on  $\text{He}^3$  and  $\text{He}^4$  obtained in various studies are related through the transformation of the  $T$  and  $n$  scales. The reduced specific heat  $C^*$ , reduced entropy  $S^*$ , and free energy  $F^*$  are, in fact, measured experimentally

$$C^* = \frac{\partial K^*}{\partial \tau} = \tau \frac{\partial S^*}{\partial \tau}, \quad F^* = K^* - \tau S^*. \quad (2)$$

To establish a relationship between the reduced parameters and the measured parameters, it is useful to choose as a thermodynamic variable instead of  $T$  the parameter  $\bar{K}$ , which is the effective temperature of the liquid, i.e., its natural characteristic. The temperature  $T$ , on the other hand, characterizes not the liquid but rather the classical heat reservoir into which it is poured. This choice is justified by the fact that the expansion of the total energy of a single atom,  $\bar{E}$ , in powers of  $\bar{K}$  can be restricted to two terms,

$$\bar{E}(n, T) = (1 + \nu)\bar{K}(n, T) + E^0, \quad \nu = \nu(n), \quad E^0 = E^0(n). \quad (3)$$

All condensed matter has a region in which relation (3) is applicable. The particular properties of condensed matter are seen in the parameter  $\nu$ . For simple crystals we have  $1 - \nu \ll 1$ ; for rigid spheres we have  $\nu = 0$ ; for quantum liquids we have  $\nu \ll 1$ . Such a division is made on the basis of the dependence of  $\bar{K}$  on the atomic mass  $m$  (Ref. 4).

$$\bar{F} = \bar{E} - T\bar{S} = (1 + \nu)\bar{K}^0 F^*(\tau) + E^0, \quad \bar{K} = -\frac{\partial \bar{F}}{\partial m} m. \quad (4)$$

Equations (3) and (4) imply a power-law dependence of  $\bar{K}^0$  on  $m$ :  $\bar{K}^0 \propto m^{-1/(1+\nu)}$ . The parameters  $\bar{K}_3$  and  $\bar{K}_4$  for liquid  $\text{He}^3$  and  $\text{He}^4$  are therefore related by

$$\bar{K}_4^0 = \bar{K}_3^0 \lambda, \quad \lambda = \left(\frac{m_3}{m_4}\right)^{1/(1+\nu)}, \quad \bar{K}_3(n, T) = \frac{1}{\lambda} \bar{K}_4(n, \lambda T). \quad (5)$$

In the  $\nu = 0$  rigid-sphere approximation, relations (5) were determined in Ref. 3. The  $\nu = 1$  approximation, i.e.,  $\lambda = \sqrt{m_3/m_4}$ , corresponds to a truly Debye amorphous solid, which helium is not. The parameter  $\nu$  is markedly different from unity even for solid  $\text{He}^3$  and  $\text{He}^4$  (Ref. 5).

There are enough relations available to relate the specific heats  $\bar{C}_3, \bar{C}_4$ , the entropies  $\bar{S}_3, \bar{S}_4$ , and the energies  $\bar{E}_3, \bar{E}_4$  of liquid  $\text{He}^3$  and  $\text{He}^4$  and, on the basis of a comparison with experiment, to determine the functions  $K^*, S^*$ , and  $C^*$ :

$$\begin{aligned} \bar{C}_3(n, T) &= \bar{C}_4(n, \lambda T), \quad \bar{C}(n, T) = (1 + \nu)C^*(\tau) \\ \bar{S}_3(n, T) &= S_4(n, \lambda T) + \ln 2, \quad \bar{S}(n, T) = (1 + \nu)S^*(\tau) + \ln g_{F, B} \\ \bar{E}_3(n, T) &= \frac{1}{\lambda} \bar{E}_4(n, \lambda T) + E^0 \left(1 - \frac{1}{\lambda}\right), \quad g_F = 2, \quad g_B = 1. \end{aligned} \quad (6)$$

The spin component of  $\ln 2$  was separated from the entropy of  $\text{He}^3$ . The dependence of  $\nu$ ,  $E^0$ , and  $\bar{K}^0$  on  $n$  is determined from the asymptotic behavior of  $\bar{C}_4$ ,  $\bar{S}_4$ , and  $\bar{E}_4$  and from the equation of state for  $P_4(n, T)$  at high  $T$ :  $P_4 = c_1 T + c_2 - c_3 T \ln T - c_4 \ln T$ ,

$$\begin{aligned} \bar{C}_4 &= (1 + \nu) \left( \frac{3}{2} - \frac{T_c}{T} \right), \quad \bar{S}_4 = (1 + \nu) \left( \frac{3}{2} \ln \frac{T}{T_s} + \frac{T_c}{T} \right), \\ \bar{E}_4 &= E^\infty + (1 + \nu) \left( \frac{3}{2} T - T_c \ln T \right). \end{aligned} \quad (7)$$

The density functions  $c_i$  are expressed in terms of the derivatives of  $\nu$ ,  $T_c$ ,  $T_s$ , and  $E^\infty$  with respect to  $n$ . For a small density  $n$  expansions (7) can be found by directly calculating the virial coefficients with the potential  $\exp -6$ . A comparison of (7) with the data of Refs. 5–8 shows that they are in agreement with the experiment within 1%, for  $10 < T < 420$  K,  $P < 10^4$  bar, and molar volume  $V > 15$  cm<sup>3</sup>/mole. Since the parameters  $c_1$  and  $c_3$  in (7) do not depend on the atomic mass  $m$ , we need to determine only two density functions  $c_2$  and  $c_4$  in any experimentally convenient region,  $T > 10$  K, to determine the equation of state for liquid  $\text{He}^3$  over a broad interval of  $n$  and  $T$ . A comparison of (6) and (7) determines the asymptotic behavior of  $K^*$ ,  $C^*$ , and  $S^*$  for  $\tau > 1$ :

$$\begin{aligned} K^* &= \frac{3}{2} \tau + \tau_0 - \tau_c \ln \tau, \quad C^* = \frac{3}{2} - \frac{\tau_c}{\tau}, \quad \tau_c = \frac{T_c}{\bar{K}^0}, \quad \tau_s = \frac{3}{2} \ln \frac{\bar{K}^0}{T_s}, \\ S^* &= \frac{3}{2} \ln \tau + \tau_s + \frac{\tau_c}{\tau}. \end{aligned} \quad (8)$$

The low-temperature asymptotic behavior ( $\tau < 1$ ) of these quantities, on the other hand, is characteristic for all nonsuperfluid and nonsolid media:

$$K^* = 1 + \Delta_1 \tau^2 - \Delta_2 \tau^4, \quad C^* = 2 \Delta_1 \tau - 4 \Delta_2 \tau^3, \quad S^* = 2 \Delta_1 \tau - \frac{4}{3} \Delta_2 \tau^3 + S_0. \quad (9)$$

For  $T_{F,\lambda} < T < \bar{K}^0$  the expansion of the specific heat in powers of  $T$ , within small effects in the statistical base of the particles, begins with the term  $\propto T$  which is not forbidden in any respect. The small residual entropy,  $S_0 \sim 0.1$ , freezes out rapidly with increasing  $\text{He}^4$  density. The plots of  $K_4^*$  and  $C_4^*$  versus  $\tau_4 = T/\bar{K}_4^0$ , constructed from the data of Refs. 6 and 9 for  $T_\lambda < T < 20$  K, are shown in Figs. 1 and 2. The labels on the curves are the values of the reduced density;  $\rho_4 = n_4/n_k$ , where  $n_k = 0.01035 \text{ \AA}^{-3}$  is the critical density of the liquid-vapor transition for  $\text{He}^4$ . This value of  $n_k$  corre-

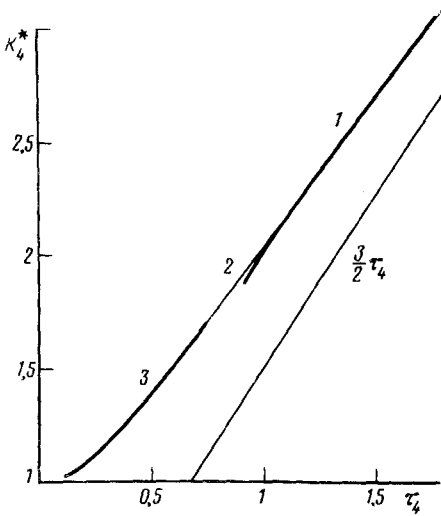


FIG. 1.

sponds to  $V_k = 58.18 \text{ cm}^3/\text{mole}$ . With increase of  $n$  and  $T$ , all the curves for  $C_4^*$  approach a single "quantum" envelope [Eqs. (8) and (9)]. Any deviation from this envelope stems from the critical fluctuations near  $T_\lambda$  and along the liquid-vapor stratification line. For the curve  $K^*$  versus  $\tau$ , on the other hand, these deviations are negligible. The reduced kinetic energy  $K^*$  can therefore be extended to the supercritical region of  $T$ , where the liquid helium is metastable. Figure 3 shows plots of  $C_3^*$  versus  $\tau_3 = T/\bar{K}_3^0$  for  $\text{He}^3$  at  $T < 2.5 \text{ K}$ , which are constructed from the data of Ref. 10. The labels on the curves are the values of the parameter  $\rho_3 = n_3/n_k$ . The situation here is the same as that in the preceding case: With increase of  $T$  and  $n_3$ , the exchange effects vanish and all the curves approach the same universal curve [Eq. (9)]—the dashed

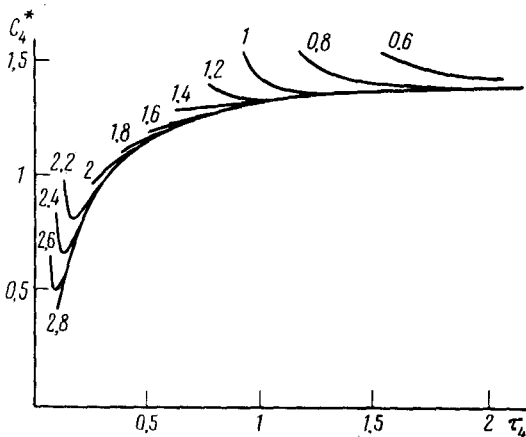


FIG. 2.



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