Physical interpretation of solutions of the theory of spontaneous compactification

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A physical interpretation is proposed for solutions of the theory of spontaneous compactification. This interpretation is based on a dimensional-reduction method.

Let us examine the relationship between dimensional reduction[1] and spontaneous compactification.[2–4] We will make use of the results of Ref. 5, where a study was made of gauge theories in multidimensional spaces of the type $M = M^4 \times G/H$ (M^4 is Minkowski space), which are symmetric under the canonical action of G on G/H. It was shown in those studies that if G/H is a symmetric space, the reduced theory in M^4 contains a single irreducible multiplet of scalar fields with a Higgs potential, which leads to a spontaneous symmetry breaking. In the present letter we show that multidimensional field configurations corresponding to a Higgs vacuum of the reduced theory satisfy the multidimensional Einstein-Yang-Mills equations and lead to a spontaneous compactification.

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We adopt a standard action with a Λ term for the gravitational field g_{MN} $[M=(\mu,m), N=(\nu,n); \mu, \nu=0,1,2,3; m, n=1,2,...,d]$ which is interacting with a gauge field A_M in a $(4+d)$-dimensional space M. A variation of this action with respect to g_{MN} and A_M yields multidimensional Einstein-Yang-Mills equations. As usual in the theory of spontaneous compactification, we seek solutions of these equations which correspond to a factorization of the space $M, M=M^4\times G/H$, with a metric of the type $g=\eta\oplus\gamma[\eta_{\mu\nu}=\text{diag}(-1,1,1,1)$ is the Minkowski metric, and γ is a G-invariant metric on $G/H]$ under the assumption $A_\mu=0$, $A_m=A_m(\xi)$, where $\xi=\{\xi^m\}$ are coordinates in the d-dimensional space G/H. As a result, we find the equations of spontaneous compactification[2-4]:

$$R_{mn} = -\frac{4\pi G}{g^2}\,\mathrm{Tr}\,(F_{ml}F_n^{\;l}); \tag{1a}$$

$$\nabla_m F^{mn} + [A_m, F^{mn}] = \partial_m F^{mn} + \Gamma^l_{ml}F^{mn} + [A_m, F^{mn}] = 0. \tag{1b}$$

We will discuss solutions of Eqs. (1b) for the case of a symmetric space G/H. The validity of Einstein equations (1a) in this case follows from the uniqueness, within a constant factor, of the G-invariant metric γ on the G/H and can be verified quite simply.[2-4]

We assume that the gauge group of the field A_M, K, and the symmetry group G are compact, simple, classical Lie groups. We denote by $\mathfrak{k}$, $\mathfrak{g}$, and $\mathfrak{h}$ the Lie algebras of groups K, G, and H; $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ is an orthogonal decomposition with respect to the canonical G-invariant scalar product $\langle,\rangle$ in $\mathfrak{g}$. We choose in $\mathfrak{g}$ a basis $\{a_{\hat\alpha}\}$ in such a way that we have $\langle a_{\hat\alpha}, a_{\hat\beta}\rangle=-\delta_{\hat\alpha\hat\beta}$ and $a_{\hat\alpha}=\{a_{\bar\alpha}, a_\alpha\}$, where $a_{\bar\alpha}\in\mathfrak{h}$ $(\bar\alpha=1,2,...\dim H)$ and $a_\alpha\in\mathfrak{m}$ $(\alpha=1,2,..., d=\dim G/H)$. It was shown in Refs. 1 that G, a symmetric field A_M on M, determines a homomorphism $\tau: H\to K$ (and a corresponding homomorphism of Lie algebras $\tau:\mathfrak{h}\to\mathfrak{k}$) and is described by the field pair $[A_\mu(x), \phi(x)]$ on M^4, where $A_\mu(x)$ is a gauge field with a definite gauge group $C\subset K$, and $\phi(x)$ is the linear mapping $\phi(x):\mathfrak{m}\to\mathfrak{k}$, which has the property

$$[\tau(a), \phi(x)(b)] = \phi(x)([a,b]), \quad a\in\mathfrak{h}, \quad b\in\mathfrak{m} \tag{2}$$

and which describes scalar fields.

The canonical action for the gauge field A_M in the multidimensional space $M=M^4\times G/H$ reduces, after a dimensional reduction (an integration over the orbit of G/H), to an action in M^4 for the gauge field $A_\mu(x)$, which is interacting with a Higgs field[1,5]:

$$S_f = \frac{1}{8g^2}\int d^4x\,\{\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) + \sum_\alpha \mathrm{Tr}(D_\mu\phi(a_\alpha)D^\mu\phi(a_\alpha)) - \lambda(|\phi(x)|^2-\kappa)^2 - V_0\}, \tag{3}$$

where $|\phi(x)|^2=-\Sigma_\alpha\,\mathrm{Tr}[\phi(x)(a_\alpha)\phi(x)(a_\alpha)]$, and λ, κ, and V_0 are constants determined by the groups H and K and by the dimension L of space G/H. It can be seen from (3) that the simplest extremals of S_f are the following field configurations:

$A_\mu = 0$ and ϕ is independent of x: and either $\phi = 0$ (an unstable maximum of the Higgs potential) or $|\phi|^2 = \kappa$ (a stable Higgs vacuum). Clearly, these configurations satisfy the Yang-Mills equations in the space $M = M^4 \times G/H$ with the metric $g = \eta \oplus \gamma$, which is the case $A_\mu = 0$, $\phi = \text{const}$ reduce exactly to the equations of spontaneous compactification, Eq. (1b).

We now wish to construct a multidimensional field $A_m(\xi)$ which corresponds to these configurations. For this purpose, we specify near point $[H]$ of the space G/H the cross section σ of the principal stratification, $G = P(G/H,H)$, and in the standard way we determine the 1-form θ, with the following value in Lie algebra $\mathfrak{g}$:

$$\theta = \sigma(\xi)^{-1} d\sigma(\xi) = \theta_{\mathfrak{h}} + \theta_{\mathfrak{m}}, \quad \theta_{\mathfrak{h}} = \theta^{\bar{\alpha}} a_{\bar{\alpha}}, \quad \theta_{\mathfrak{m}} = \theta^\alpha a_\alpha .$$

This form satisfies the Maurer-Cartan equation $d\theta^{\hat{\alpha}} = -\frac{1}{2} C^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}} \theta^{\hat{\beta}} \Lambda \theta^{\hat{\gamma}}$, where $C^{\hat{\alpha}}_{\hat{\beta}\hat{\gamma}}$ are structure constants of the group G (Refs. 3, 4, and 6). For the form $\theta_{\mathfrak{m}} = \theta^\alpha_m a_\alpha d\xi^m$ we introduce a dual basis of vector fields, $\{\theta^m_\alpha\}$, on G/H, which satisfies the conditions $\theta^m_\alpha \theta^\beta_m = \delta^\beta_\alpha, \theta^m_\alpha \theta^\alpha_n = \delta^m_n$.

A multidimensional gauge field corresponding to these four-dimensional configurations is specified by the following 1-form on G/H:

$$A = \tau(\theta_{\mathfrak{h}}) + \phi(\theta_{\mathfrak{m}}), \tag{4}$$

where ϕ is independent of x and gives us an extremum of the potential in (3). This formula is the most general representation for G of a symmetric gauge field of the necessary type (see Van's theorem in Chapter X in Ref. 6). An ansatz with $\phi = 0$ has been used in most of the studies, of which we are aware, in which compactified solutions have been constructed (see Refs. 2–4, for example). The G-symmetric field in the general form in (4) was examined in Ref. 7. The stress tensor of the gauge field corresponding to (4) is

$$F_{mn} = F_{\alpha\beta} \theta^\alpha_m \theta^\beta_n, \quad F_{\alpha\beta} = [\phi(a_\alpha), \phi(a_\beta)] - \tau([a_\alpha, a_\beta]) . \tag{5}$$

We will now show that the gauge field which is specified by form (4), and which corresponds to extrema of functional (3), satisfies not only Eq. (1b) but also the stronger condition of parallelism:

$$\nabla_k F_{mn} + [A_k, F_{mn}] = \partial_k F_{mn} - \Gamma^l_{km} F_{ln} - \Gamma^l_{kn} F_{ml} + [A_k, F_{mn}] = 0 . \tag{6}$$

To show this, we note that the G-invariant canonical metric on G/H is specified by the formula $\gamma_{mn} = L^2 \theta^\alpha_m \theta^\alpha_n$. The Christoffel symbols of the Riemannian connection here are

$$\Gamma^m_{nk} = \frac{1}{2}\left\{ C^\gamma_{\bar{\beta}\alpha} \theta^m_\gamma (\theta^{\bar{\beta}}_k \theta^\alpha_n + \theta^{\bar{\beta}}_n \theta^\alpha_k) + \theta^m_\alpha \left(\frac{\partial \theta^\alpha_n}{\partial \xi^k} + \frac{\partial \theta^\alpha_k}{\partial \xi^n} \right) \right\} . \tag{7}$$

Substituting (4), (5), and (7) into (6), and using condition (2) and the Bianchi identity for the structure constants of group G, we find that parallelism condition (6) reduces to the following condition on the mapping ϕ:

$$[\phi(a_\alpha), [\phi(a_\beta), \phi(a_\gamma)]] - \phi([a_\alpha, [a_\beta, a_\gamma]]) = 0. \tag{8}$$

Making use of the explicit form of mapping[5] ϕ, we see that (8) holds for the case $|\phi|^2 = \kappa$; in the case $\phi = 0$, it holds trivially.

We note in conclusion that the results of this letter actually constitute a new approach to a physical interpretation of the solutions of a theory of spontaneous compactification. This new interpretation is based on a comparison of the vacuum states of multidimensional and reduced theories. It turns out that from this point of view the field configurations of nearly all presently known compactified solutions of the Einstein-Yang-Mills equations (Ref. 7 is an exceptional case) correspond not to a Higgs vacuum of the reduced theory but to an unstable local maximum of its Higgs potential. We believe that this interpretation of the solutions of the theory of spontaneous compactification reveals their physical meaning and should be taken into consideration in any formulation of a problem in this theory.

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