String theory and structure of universal modulus space

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A sum over all loops in string theory in the limit of an infinite coupling constant can be regarded as a new two-dimensional field theory.

1. One of the basic problems in string theory today is that of formulating a first quantization outside the framework of an expansion in loops. Expressions have been derived for the amplitudes for the scattering of string states for an arbitrary number of loops, p. These amplitudes are expressed in terms of finite-multiplicity integrals over the module spaces M_p of Riemannian surfaces of type p with known integrands,[1] which are proportional to the Mumford measures $d\mu_p$. The perturbation-theory series for the amplitude A is written

$$A \sim \sum_{p=0}^{\infty} g^p \int_{M_p} a_p d\mu_p . \qquad (1)$$

It would be very useful to have a formulation of the theory in which the integration is carried out over a single "universal modulus space" which contains all the spaces M_p (Ref. 2, for example). The dimensionalities of the spaces M_p are different: $\dim_C M_p = 3p-3$ for $p \geqslant 2$. The modulus spaces M_p with $p < p'$ fit naturally into the space $M_{p'}$; they lie on the boundary of $M_{p'}$. Accordingly, one might select as a universal modulus space the infinite-dimensional space $M_\infty = \lim_{p\to\infty} M_p$. If M_p is nested in $M_{p'}$, the measure $a_p d\mu_p$ agrees with the restriction of $a_{p'} d\mu_{p'}$ on M_p. This condition fixes the relative normalization of a_p for various p.

When expression (1) is written in terms of the universal modulus space, it should take the form

$$A \sim \int_{M_\infty} a_\infty \, d\mu_\infty (g). \qquad (2)$$

This integral over an infinite-dimensional space can be written as a path integral in some field theory (more on this below). For finite coupling constants g, the measure $d\mu_\infty(g)$ must be highly singular: Only in this case will the finite-dimensional subspaces M_∞ make a finite contribution to the integral. From the standpoint of the universal modulus space, a string theory with $g = \infty$ is special; the dynamics of such a theory would be determined entirely by surfaces of an infinite type. (We recall that in superstring theory the Yang-Mills coupling constant is expressed in the low-energy limit in terms of the product of g and the vacuum expectation value of a dilation field. It is difficult to find a finite vacuum expectation value of a dilaton field, so that the limit $g = \infty$ bears directly on realistic string models.) This consideration determines the last unfixed parameter in anomaly-free string theories. This model appears to be the simplest string model.

2. A universal modulus space can be described in several ways. The representation which has been developed most extensively—in the form of an infinite-dimensional grassmannian Gr—arose originally in the theory of Kadomtsev-Petviashvili equations.[3–5] This representation is related to the operator formalism, which has recently been developed for describing loop corrections in string theories.[6,7] The measures in which we are interested, $a_p\, d\mu_p$ and $a_\infty\, d\mu_\infty$, are the cross sections of certain stratifications (or, more precisely, sheafs) on the spaces M_p and M_∞, respectively. They are constructed from the stratifications of j-differentials L_j on Riemannian surfaces. For simplicity, we restrict the analysis to the case of L_0: functions on a Riemannian surface. On surface X we note the point ξ and a small circle S around this point. On the surface there is a single function which is holomorphic everywhere: a constant. All other analytic functions necessarily have poles, i.e., are meromorphic. We consider a set of functions which are holomorphic away from ξ, i.e., which have poles only at ξ. This set of functions is defined unambiguously on surface X, for otherwise the surface could be reconstructed from this set itself. The restriction of the functions of this set to the circle S singles out a certain linear subspace $W_{X\xi}$ in the space H of all functions on the circle. For example, if X is a Riemannian sphere (type $p=0$), and if ξ is its north pole, then the set of functions which are holomorphic away from ξ are linear combinations of the functions $1, z, z^2, \ldots$, and in the linear space H of all functions on a circle with a basis $\{t^k\}$ ($t=e^{i\phi}$, where k is an integer) a linear subspace H_+, which is constructed on the basis vectors t^k with $k \geqslant 0$. For surfaces X of a higher type, the subspaces W depend holomorphically on the complex structure on X—on a point in modulus space M_p. For example, on a torus ($p=1$), which is conformally equivalent to a parallelogram $(1, \tau)$, functions which are holomorphic away from ξ are constructed in terms of the Weierstrass function,

$$P(z \mid \tau) = \left[\frac{\theta'_*(0)}{\theta_*(z)}\right]^2 \sum_e \frac{\theta_e^2(z)}{\theta_e^2(0)} = 1/z^2 + p_2(\tau)z^2 + p_4(\tau)z^4 + \ldots,$$

and its derivatives: $1, P(z-\xi\,|\tau); -\frac{1}{2}P'(z-\xi\,|\tau); \frac{1}{6}P''(z-\xi\,|\tau), \ldots$ (a first-order pole is forbidden). The subspace W in H which is associated with the torus $(1,\tau)$ is constructed on the basis

$$\Big\{1;\ t^2 + p_2(\tau)t^{-2} + p_4(\tau)t^{-4};$$

$$t^3 - p_2(\tau)t^{-1} - 2p_4(\tau)t^{-3} - \ldots; \ldots;\ t^{2n+2} + \frac{1}{2n+1}p_{2n}(\tau)$$

$$+ (n+1)p_{2n+2}(\tau)t^{-2} + \ldots; \ldots\Big\}, \left(t = \frac{1}{z-\xi}\right),$$

which depends on τ. In general, nearly all linear subspaces W in H, whose orthogonal projections onto $H_+ = \{t^k, k \geqslant 0\}$ have a finite kernel and a finite cokernel, are associated unambiguously with Riemannian surfaces X with our point ξ (and a coordinate system near ξ).[4] For example, the type p of the surface X is determined by the circumstance that W has no vectors of the type $t^1[1+0(t^{-1})]$; $t^2[1+0(t^{-1})]$;...$t^p[1+0(t^{-1})]$ (this assertion is correct for nearly all surfaces and selected points; the exact criterion is slightly more complicated). In transforming from the functions L_0 to arbitrary j-differentials, we must replace the word "function" by "stratification cross section L_j" everywhere. The collection of cross sections L_j which are holomorphic away from ξ naturally depends on the surface, its complex structure, the selected point, and j. In particular, the index of the orthogonal projection $W \rightarrow H_+$ (the difference between the dimensionalities of the kernel and cokernel) is $\chi(L_j) - 1 = (2j-1)(p-1) - 1$. For this reason, it is convenient to examine the projection of W not onto H_+ but onto the subspace $H_+^{(j)} = \{t^k, k \geqslant 1 - (2j-1)(p-1)\}$. The index is then zero.

We thus see that the collection of sets W (which is equal to a collection of linear subspaces in H and which is also equal to an infinite-dimensional grassmannian Gr) contains in itself a universal modulus space: $M_\infty \subset \mathrm{Gr}$. (Gr also contains a dependence on the selected point ξ.) For the amplitude A at $g = \infty$ there must exist a representation in the form of an integral over Gr with a nonsingular measure:

$$A = \int_{Gr} a\,d\mu\,. \tag{3}$$

Actually, this formula can be written as a path integral of a two-dimensional field theory. Each point of grassmannian W is a collection of basis vectors $\Sigma_{k \leqslant s} c_k^{(s)} t^k$ with various s. We introduce yet another variable, u. All the information on W is thus contained in the function of two variables $W(t,u) = c_k^{(s)} u^s t^k$. The grassmannian itself is a collection of fields $W(t,u)$ and it must be possible to write integral (3) in the form

$$A = \int DW\, \Phi_A\, \{W(t,u)\}\,. \tag{4}$$

Any dependence on the choice of amplitude A can be dealt with by introducing a source $J(v) = \Sigma_{n=1}^{\infty} x_n v^n$:

$$A = \Gamma_A\, \{J(v)\} \int DW \Phi\, \{\, W(t,u);\; J(v)\,\}\,. \tag{5}$$

3. So far, we do not know the explicit form of functional Φ in (5). It is clear, however, that it must be expressed in terms of so-called τ-functions on the grassmannian and the related entities: the σ-cross sections of determinant stratifications (Ref. 4, for example). It must also have certain covariance properties. The covariance must provide independence from the choice of the point ξ, the coordinates near it, and the choice of the basis in space H. [The latter requirement is analogous to the condition of $SO(n)$ covariance for finite-dimensional grassmannians: $SO(n)/SO(k) \times SO(n-k)$.] Another requirement is that on finite-dimensional subspaces M_p in Gr the measure in (4) be factorized into known loop formulas[1] and the measure in a subspace orthogonal to M_p.

The τ-functions are special functions which depend on the W grassmannian point and the source J. They are the determinants of an orthogonal projection of the linear space $e^{J(t)}W$ obtained from W by means of a linear transformation with an infinite-dimensional matrix $e^{J(t)}$ onto $H_+^{(J)}$: $\tau_W^{(j)}\{J\} = \det[e^{J(t)}W \to H_+^{(j)}]$. According to this definition, the τ-functions carry information on the value of j. The meaning of representation (5) is that the action of the standard operators of the Virasoro algebra, $\Gamma(z) = \exp\sum_{n=1}^{\infty} z^n x_n \exp\sum_{n=1}^{\infty} z^n(\partial/\partial x_n)$, on a τ-function generates the Green's functions of j-differentials on the Riemannian surface:

$$[\Gamma(z)\overset{*}{\Gamma}(z')\tau_W^{(j)}\{x_n\}]\big|_{x_n=0} \;/\; \tau_W^{(j)}(0) = G_W^{(j)}(z,z'). \tag{6}$$

(In the case of 1/2-differentials, this assertion can be related to the formulas of Ref. 5 for τ-functions. Those formulas represent these functions as path integrals along free fermions on a circle: $\tau_W\{x_n\} = \int D_{\tilde\psi} D_\psi e^{\tilde\psi C_W \psi} e^{\Sigma_n x_n \Lambda_n(\tilde\psi,\psi)}$.) The operator C_W must be constructed in such a way that for W corresponding to the Riemannian surface X we have $X,\tilde\psi C_W \psi = \int_X \tilde\Psi\bar\partial\Psi$, where Ψ are 1/2-differentials on X. (For the other J-differentials we can use the "fermionization formulas" of the type discussed in Ref. 8, which express their Green's functions in terms of the correlation functions of the 1/2-differentials.) The validity of (6) for 1/2-differentials has already been pointed out in Refs. 5 and 7. Alvarez-Gaumé *et al.*[7] also pointed a relationship among the τ-functions corresponding to different j-differentials.

To find the functional Φ ("action" of the two-dimensional field theory), we need to know not only the Green's functions but also the determinants of the operators $\bar\partial_j$ and the metrics in the determinant stratifications. In order to approach this problem more directly, it would appear to be necessary to use the known covariance properties of the unknown measure on the grassmannian.

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