Formation of stability regions from unstable states in dissipative nonlinear systems

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A method is proposed for finding the asymptotic form of the solutions of second-order equations which describe a transition from an unstable initial state to a stable state.

Many problems from various branches of physics reduce to the problem of describing the nonlinear stage of an instability. Such problems arise if an initial state, which is exponentially unstable in the linear approximation, is acted upon by a system of external fields, which send it from a stable state to an unstable state rapidly (in comparison with the typical growth rate). The basic problem of nonlinear dynamics here is to determine how the system relaxes to new, stable states, if such exist. There is no hope of finding an analytic solution for the problem in such a general formulation, since the nature of the decay of the unstable state depends strongly on the initial conditions, which may by uncontrollable (of a fluctuational nature).

For the case of localized initial perturbations, however, Kolmogorov, Petrovskiĭ, and Piskunov[1] have proved an exceedingly strong assertion regarding the nature of the evolution of such initial data for a nonlinear diffusion equation (a typical equation of this type is the well-known Ginzburg-Landau equation). Specifically, they proved that for systems of this type a transition from an unstable steady-state solution to a stable steady-state solution continues a long time by means of stationary travelling waves, whose velocity is determined unabiguously by the linear part of the equation, while the

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shape of the front is determined by the solution of the corresponding ordinary differential equation. Many current studies are based to some extent on the fundamental results of Ref. 1 (see Ref. 2 and the bibliography there).

In the present paper we show that results of a similar type hold for a wider range of problems. In addition, we find that for sufficiently small initial conditions it is possible to relate the temporal asymptotic behavior of the solutions to the data of a Cauchy problem. This is a particularly interesting circumstance, since we are dealing with equations which are not integrable.

We will demonstrate the validity of the assertions above in two specific physical examples.

The dynamics of a Fréedericksz transition in nematic liquid crystals in the presence of return flows is described in terms of dimensionless variables by an unstable version of the sine-Gordon equation,

$$u_{tt} - u_{xx} + 2\gamma u_t = \sin u, \quad u \to 0 \quad \text{as} \quad |x| \to \infty , \tag{1}$$

where $u = 2\varphi$ (φ is the angle through which the director deviates from its initial position), and $\gamma > 0$ is proportional to the effective viscosity.

Our task is to study the dynamics of the transition from the perturbed solution $u = 0$ to the "stable vacuums" $u = \pi$ (mod2π). We assume that the initial data for Eq. (1) are concentrated on a finite interval and are quite small.[1)] For small values of $u(x, t)$, Eq. (1) can be replaced by its linear version, whose solution is given by the integral

$$u(x, t) = \int_\Gamma \{C^+(k)\exp(-i\omega_+ t) + C^-(k)\exp(-i\omega_- t)\}\, e^{ikx}\, dk , \tag{2}$$

where $\omega_\pm = -\gamma \pm i(k^2 - 1 - \gamma^2)^{1/2}$, and the contour Γ passes through the upper k half-plane from $-\infty$ to $+\infty$. It is convenient to choose the branch of the root in such a way that at large values of $|k|$ it is equal to k.

Let us examine the behavior of $u(x, t)$ in (2) at large values of x and t: $x = Vt + \tilde{x}$, $t \to \infty$, where $V, \tilde{x} = O(1)$. The analytic properties of the integrand in (2) have the result that at sufficiently large values of t and $|V| > 1$ the quantity $u(x, t)$ vanishes identically. At $v^2 < 1$, integral (2) has a saddle point $k = ik(V)$, which determines its asymptotic behavior as $t \to \infty$:

$$u(x, t) = (2\pi)^{1/2} t^{-1/2} (1+\gamma^2)^{1/4} (1-V^2)^{-3/4} C^+\left(iV\sqrt{\frac{1+\gamma^2}{1-V^2}}\right)$$

$$\times \exp\left[(-\gamma + \sqrt{(1-V^2)(1+\gamma^2)})t - \tilde{x}\sqrt{\frac{1+\gamma^2}{1-V^2}}\right] . \tag{3}$$

The exponent in (3) increases over time if $V^2 < V_c^2 = (1+\gamma^2)^{-1}$ and decreases if $V^2 > V_c^2$.

If the initial condition (i.e., actually C^+) is so small that $u(x, t)$ in (3) nevertheless is quite small for some time within the range of applicability of the method of steepest decent, then (3) gives us the solution of (1). Here it can be assumed that at $|V| > V_c$ we have $u(x, t) = 0$ as $t \to \infty$. At $|V| < V_c$, however, solution (3) quickly becomes inapplicable.

Let us examine the case $V = V_c$ in more detail. Here we have

$$u(x, t) = (2\pi)^{1/2}\gamma^{-3/2}(1+\gamma^2)C^+(i\gamma^{-1}\sqrt{1+\gamma^2})t^{-1/2}\exp[-(1+\gamma^{-2})^{1/2}\tilde{x}] \quad (4)$$

and this quantity is small in the limit $t \to \infty$ if $\tilde{x} \simeq O(1)$. This solution corresponds to the one which we have been seeking if we can continue it in a reasonable way into the region of large negative values of $\tilde{x}$, where (4) obviously does not hold. The time dependence of the coefficient of the exponential function in (4) is slow in comparison with $\exp(-V_c t)$, so that $u(x, t)$ depends primarily on $\tilde{x} = x - V_c t$.

Equation (1) has a solution of the type $U(z)$, where $z = (x - V_c t)\sqrt{1+\gamma^{-2}}$, if $u(z)$ satisfies the ordinary differential equation

$$u_{zz} + 2u_z + \sin u = 0. \quad (5)$$

It's easy to see that solutions (5), which are bounded at all z, can have only the following asymptotic behavior: $u \to 0$ as $z \to +\infty$ and $u \to \pm\pi$ as $z \to -\infty$. For Eq. (1), these solutions describe steady-state travelling waves which are propagating at a velocity V_c into the unstable region, $u = 0$. Behind the wavefront, a stable region, $u = \pm\pi$, forms. We will show that such solutions join well with linear equation (4).

We fix the solution of (5) in the following way:

$$U \to \pi \quad \text{as} \quad z \to -\infty; \quad U = e^{-z} + Aze^{-z} \quad \text{as} \quad z \to +\infty. \quad (6)$$

The constant A is determined unambiguously [we will use $U(z)$ below to represent specifically this solution]. We now consider the asymptotic behavior of the solution $U(z+b)$ as $b \to \infty$, $z = O(1)$. From (6) we find

$$U(z+b) = Abe^{-z}e^{-b}(1 + O(b^{-1})). \quad (7)$$

Comparing (4) and (7), we see that the parameter b should be determined from the equation

$$Abe^{-b} = t^{-1/2}(2\pi)^{1/2}\gamma^{-3/2}(1+\gamma^2)C^+(i\gamma^{-1}\sqrt{1+\gamma^2}).$$

As a result, we find the solution

$$u(x, t) = U(z + b(t)) \quad (8)$$

as the leading term of the asymptotic behavior of the solution of Eq. (1) in the region $0 < V < 1$. A solution for a wave travelling to the left can be derived in a corresponding way.

As a second example we consider the equation

$$u_{xt} + u_{tt} + 2\sigma u_t = \sin u, \quad u \to 0, \ x \to \infty, \quad (9)$$

which arises in a simple version of the theory of laser amplifiers,[4,5] in which the conductivity of the medium, σ, at the transition frequency is taken into account.[2] The corresponding formulation of the problem for (9) is $u(x,0)=0$ at $x>0$, $u(0,t)=u(t)$; $u(t)=0$ at $t<0$. This problem corresponds to a Cauchy problem in x.

As in the preceding example, we assume that the incoming pulse $u(t)$ is weak. In this case, the linear approximation holds in a certain region:

$$u(x,t) = \int_{-\infty}^{\infty} \tilde{u}(\omega)\exp\left[i\omega(x-t) + \frac{ix}{\omega} - 2\sigma x\right]d\omega\ , \qquad (10)$$

where $\tilde{u} = (2\pi)^{-1}\int_0^\infty u(t)\exp(i\omega t)dt$.

Let us examine the asymptotic behavior of (12) as $x,\ t\to\infty$, $t = xV^{-1}+\tilde{t}, V,\tilde{t}\sim O(1)$. For $0<V<1$, integral (10) can again be evaluated by the method of steepest decent:

$$u(x,t) = (2\pi)^{1/2}t^{-1/2}\left(\frac{V}{1-V}\right)^{3/4}\tilde{u}\left[i\left(\frac{V}{1-V}\right)^{1/2}\right]$$
$$\times\exp\left\{2x\left[\left(\frac{1-V}{V}\right)^{1/2} - \sigma\right]\right\}\exp\left[\tilde{t}\left(\frac{V}{1-V}\right)^{1/2}\right]. \qquad (11)$$

Expression (11) falls off exponentially with increasing x at $V>V_c=(1+\sigma^2)^{-1}$ and increases at $V<V_c$. At $V=V_c$ we find from (11)

$$u(x,t) = (2\pi)^{1/2}t^{-1/2}\sigma^{-3/2}\tilde{u}(i\sigma^{-1})\exp(\tilde{t}\sigma^{-1})\ . \qquad (12)$$

The approximation is thus applicable in the limit $t\to\infty$ if $V>V_c$ and also at $V=V_c$, $\tilde{t}=O(1)$ [in this case, $u(x,t)$ depends substantially on only $\tilde{t}=t-xV_c^{-1}$].

To find an asymptotic solution of (9), we must again find a solution of (9) of the type $u(t-xV_c^{-1})$, which would join with (12). Assuming $z=-\tilde{t}\sigma^{-1}$, we see that the problem reduces entirely to the preceding problem: $u(z)$ satisfies (5). Joining $U(z+C)$ as $C\to\infty$ with (12), we find $AC\exp(-C)=(2\pi)^{1/2}t^{-1/2}\sigma^{-3/2}\tilde{u}(i\sigma^{-1})$, so that the asymptotic behavior of $u(x,t)$ as $x\to\infty$ is

$$u(x,t) = U\left[\frac{x-V_c t}{\sigma V_c} + C(t)\right].$$

In summary, we have shown that a transition from an unstable state to a stable state for Eqs. (1) and (9) occurs asymptotically in the form of a travelling wave. We have proposed a method for finding the shape of the wavefront and its velocity. The results derived here constitute an analog of a theorem formulated in Ref. 1. This method will be generalized to the multidimensional case and (possibly) other classes of equations in a separate study.

[1]This restriction apparently has no influence on the qualitative nature of the results.

[2]An asymptotic theory for long amplifiers without dissipation ($\sigma=0$) was studied in Refs. 6 and 7.

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