Chiral boson anomaly in a gravitational field

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The axial photon current $K^\mu = -e^{\mu\nu\kappa\lambda}A_\nu\partial_\kappa A_\lambda$ is found to exhibit in an external gravitational field the triangle anomaly which is similar to the anomaly for the axial fermion current.

Soon after the discovery[1] of the axial-current anomaly in an external electromagnetic field, a similar anomaly was detected[2] in an external gravitational field, so that the axial-current divergence a^μ of a massless Dirac spinor is

$$\partial_\mu a^\mu = \frac{\alpha}{2\pi}F_{\mu\nu}\tilde{F}^{\mu\nu} - \frac{1}{192\pi^2}R_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} . \tag{1}$$

Here $F_{\mu\nu}$ is the stress tensor of the electromagnetic field, $R_{\mu\nu\alpha\beta}$ is the Riemann tensor, and $\tilde{F}^{\mu\nu}$ and $\tilde{R}^{\mu\nu\alpha\beta}$ are their dual quantities

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}, \qquad \tilde{R}^{\mu\nu\alpha\beta} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta} .$$

It should first be noted that the gravitational anomaly may be regarded as a kind of spin effect. Let us assume that $R\tilde{R} \neq 0$ in a certain region of space. According to (1), the conservation of not only the axial current a_μ but also the lepton-charge current $j_\mu = \frac{1}{2}(v_\mu + a_\mu)$ of the Weyl neutrons (v_μ is the Dirac vector current) will then be violated in this region. In other words, this region gives rise to a lepton-charge flux. Since the gravitons do not interact directly with the lepton charge, this behavior is surprising. This paradox can be resolved by taking into account that in the case of neutrinos the lepton charge is uniquely related to the chirality. Relation (1) can then be interpreted in a natural way as the result of a spin polarization of vacuum: The particles of a given chirality are attracted to the "center"—the source of the field $R_{\mu\nu\alpha\beta}$, while the particles of the opposite chirality are emitted.

If this interpretation is correct, a similar anomaly will also be exhibited by the boson fields with a spin—the spin effects in the gravitational field are universal.

In this letter we assert that this anomaly does indeed exist and that for photons it is described by the relation

$$\partial_\mu K^\mu = -\frac{1}{96\pi^2}R_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} , \tag{2}$$

where $K^\mu = -e^{\mu\nu\alpha\beta}\partial_\alpha A_\beta$, and A_ν is the vector potential of the electromagnetic field. Because the operators are identical, $\partial_\mu K^\mu = -\frac{1}{2}F\tilde{F}$ in (2) can be written in the form

$$F\tilde{F} = \frac{1}{48\pi^2}R\tilde{R}. \tag{3}$$

The sense in which relation (2) is a complete analog of relation (1) can be explained as follows. First, a brief comment on the choice of the current K^μ. Since an infinitesimal mass can be introduced under any circumstance when dealing with an anomaly, we will first consider the case $m_\gamma \neq 0$. Both the current K^μ for photons and the current $a^\mu = \bar{\psi}\gamma^\mu \gamma_5 \psi$ for fermions will then be a Pauli-Lubansky vector.[3] Now, in the limit $m_\gamma \to 0$ the mean value of the operator $\int K^0 d^3x$ is 1 for a counterclockwise-polarized photon and -1 for a clockwise-polarized photon under the condition that the wave function of the photon is normalized in a standard way. The "charge" $\int K^0 d^3x$ thus measures the difference between the number of left and right photons.

The current K^μ is not a gauge-invariant current. It is easy to see, however, that since the charge $\int K^0 d^3x$ is invariant under local gauge transformations, it can be used to classify the states in the perturbation theory.

Let us now consider the conversion of the current a^μ into two photons and two gravitons and the conversion of the current K^μ into two gravitons. Each of these conversions is described by a single form factor

$$
\begin{aligned}
\langle 0 | a^\mu | 2\gamma \rangle &= f_1(q^2)\, q^\mu F\tilde{F} \\
\langle 0 | a^\mu | 2g \rangle &= f_2(q^2)\, q^\mu R\tilde{R} \\
\langle 0 | K^\mu | 2g \rangle &= f_3(q^2)\, q^\mu R\tilde{R},
\end{aligned} \tag{4}
$$

where q is the current momentum. A naive assumption would be that the imaginary parts of the form factors f_1, f_2, and f_3 vanish for the photon current K^μ and for the massless fermions. In particular, the imaginary part $\mathrm{Im} f_i(q^2)$ should vanish by virtue of the conservation of chirality in the electromagnetic and gravitational interactions. The anomaly in this sense consists in that the imaginary part is actually nonzero and that it is proportional to the δ-function, $\delta(q^2)$. To see this, we need only to add, for example, a small mass to the intermediate particles and then let it tend to zero. The dispersion integral of the imaginary part remains constant even as $m \to 0$. A similar analysis of the triangle anomaly for fermions in an external electromagnetic field was carried out by Dolgov and Zakharov.[4] Here we generalize it for the photon triangle in an external gravitational field.

Calculations yield the following results:

$$
\begin{aligned}
\mathrm{Im} f_1(q^2) &= -\frac{1}{4q^2}(1-v^2)\ln\frac{1+v}{1-v}, \\
\mathrm{Im} f_2(q^2) &= \frac{1}{128\pi}\frac{1}{q^2}(1-v^2)^2\ln\frac{1+v}{1-v}, \\
\mathrm{Im} f_3 &= \frac{1}{32\pi}\frac{1}{q^2}v^2(1-v^2)\ln\frac{1+v}{1-v},
\end{aligned} \tag{5}
$$

where v is the c.m. velocity. Determining then, by means of the dispersion relations, the real part of the form factors $f_{1,2,3}$ from their imaginary part, we find (1) and (2) in the limit $m \to 0$.

The chiral boson anomalies were discussed previously for the antisymmetic tensor potential $A_{\mu\nu}$ in a gravitational field[5] and for the gluon current K_μ in an external Yang-Mills field.[6] All known chiral anomalies may be regarded as anomalies in the Pauli-Lubansky vector. It is now clear that gravitons in an external gravitation field also have such an anomaly. We can assume that the coefficients of the anomaly are proportional to the square of the helicity (1:4:16 for a Weyl spinor, photon, and graviton, respectively).

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