Fluctuations of the local state density in a one-dimensional conductor

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The distribution of the local state-density fluctuation in a one-dimensional conductor is evaluated. In the case of a finite system with an open boundary, the distribution has the shape of a logarithmically normal law. For an infinite or shorted sample the fluctuations are described by the inverse Gaussian distribution. The generalization of the results obtained here to the case of an insulator of arbitrary dimensionality is discussed.

1. The problems associated with the statistical (mesoscopic) fluctuations in conductors with a weak disorder have recently been discussed extensively in the literature. The questions raised in connection with the distribution function of these fluctuations, questions of considerable interest, have been discussed. It was shown in Refs. 1 and 2 that fluctuations in the resistance of one-dimensional conductors satisfy a logarithmically normal distribution law. Earlier, Wegner[3] obtained for an arbitrary dimensionality $d = 2 + \epsilon$ an expression for the local state-density moments, which corresponds to the logarithmically normal asymptotic behavior of the distribution function. In his study Wegner[3] used a nonlinear σ-model in a single-loop approximation. Logarithmically normal asymptotic behavior in the distribution of the conductivity fluctuations and the total state density in the dimensionality $(2 + \epsilon)$ was also observed in the single-loop approximation.[4]

In the present letter we determine the exact distribution of the local state-density fluctuations in the one-dimensional case, which is valid for the typical values and in the region of distant tails.

2. At an energy E the standard definition of the state density at the point x is

$$\rho_0(E,x) = \sum_\mu |\psi_\mu(x)|^2 \delta(E-E_\mu) = \langle x|\delta(E-\hat{H})|x\rangle, \quad (1)$$

where the subscript μ specifies the exact eigenstates of the Hamiltonian $\hat{H}$ with the wave function ψ_μ and energy E_μ. A more general quantity can be considered, and in the case of an infinite or shorted sample must be considered. This quantity differs from (1) in that the δ functions are diffuse: the δ-functions, are replaced, for example, by a Lorentzian with a width η. An average over the scale Δ can also be taken in the coordinate space:

$$\rho_{\eta,\Delta} = \int_{-\Delta}^{\Delta} \frac{dy}{2\pi\Delta} \sum_\mu \frac{\eta}{(E-E_\mu)^2+\eta^2} |\psi_\mu(x+y)|^2. \quad (2)$$

Let us consider the case of a slight disorder $E\tau \gg 1, l \gg \lambda$, where l and τ are respectively the length and time of the mean free path of an electron, and λ is the electron wavelength. All the lengths below are measured in units of l and all energies are given in the units of τ^{-1}. We restrict the discussion here to the case $\eta \ll 1$. Without attempting to describe the situation for an arbitrary Δ, let us consider the fluctuations of two quantities, ρ and $\tilde{\rho}$, which are given by Eq. (2) for $\Delta \ll \lambda$ and for $\lambda \ll \Delta \ll 1$ (Ref. 6), respectively. The case $\Delta \gg 1$ was considered in Ref. 7.

3. To evaluate the moments of the distribution function, $W(\rho)$ and $W(\tilde{\rho})$, we make use of Berezinskii's[5] diagram technique, in terms of which these moments can be written in the form

$$\begin{Bmatrix} \langle \rho^n(x)\rangle \\ \langle \tilde{\rho}^n(x)\rangle \end{Bmatrix} = \sum_{m=0}^{\infty} m^{n-2} \begin{Bmatrix} a_n \\ \tilde{a}_n \end{Bmatrix} R_m(\eta,x) L_m(\eta,x). \quad (3)$$

Here R_m (L_m) are the blocks introduced by Berezinskii. These blocks are equal to the sum of the diagrams which are situated to the right (left) of the point x and which have $2m$ free ends at this point. Equation (3) also contains a combinatorial factor $m^{n-2}\, a_n$ or $m^{n-2}\, \tilde{a}_n$, which is determined by the number of methods used to form a block R_m from n loops. At $m \gg 1$

$$a_n = 2^{1-n} \frac{n-1}{n(2n-1)} \frac{\Gamma^2(2n)}{\Gamma^5(n)}; \qquad \tilde{a}_n = \frac{n-1}{2n-1} \frac{\Gamma(2n)}{\Gamma^3(n)}, \quad (4)$$

where $\Gamma(n)$ is the gamma function. The quantity $R_m(\eta,x)$ satisfies the equation

$$\left(8\eta m + 2\frac{d}{dx}\right) R_m = m^2 (R_{n+1} + R_{m-1} - 2R_m). \quad (5)$$

For an open system the boundary condition for (5) is $R_m(x=0) = \delta_{m0}$. To solve (5), we can use a Laplace transform with respect to x and we can write the Laplace variable in the form $q(q+1)$. In the case of relevant large values of m, for the Laplace transform of R_m we find

$$r_m(\eta, q^2+q) = \Gamma^3(q)\Gamma^{-2}(2q)(2q+1)^{-1}(8\eta)^q (2m\eta)^{1/2} K_{2q+1}(2(8m\eta)^{1/2}), \quad (6)$$

where $K_q(z)$ is a Bessel function. In a semi-infinite system we have

$$L_m(\eta, x) = R_m(\eta, x \to \infty) = q r_m(\eta, q)\Big|_{q=0} = 2(8m\eta)^{1/2} K_1(2(8m\eta)^{1/2}). \quad (7)$$

Using (3), (4), (6), and (7), we find the following expression for the moments in a semi-infinite system:

$$\langle \tilde{\rho}^n(x)\rangle = \frac{\Gamma(n+1)}{2n-1}(8\eta)^{1-n}\left\{1 - \frac{n-1}{\pi n}\int_{-\infty}^{\infty} dp \frac{(n+ip-{}^1/_2)^2}{2ip+1}\frac{\Gamma^3({}^1/_2 - ip)}{\Gamma^2(-2ip)}\right.$$

$$\left.\times \left|\frac{\Gamma(n+ip-{}^1/_2)}{\Gamma(n)}\right|^4 \exp\left[-\frac{x}{8}\left(1+2\frac{\ln 8\eta}{x}\right)^2 + \frac{x}{8}\left(2ip - 2\frac{\ln 8\eta}{x}\right)^2\right]\right\}. \quad (8)$$

4. First, we consider the case $\eta = 0$. From (8) we find

$$\langle \tilde{\rho}^n\rangle = e^{n(n-1)x/2}; \qquad W(\tilde{\rho}) = (2\pi x)^{-1/2}\frac{1}{\rho}e^{-(1/2x)(\ln\tilde{\rho} + x/2)^2}; \quad (9)$$

i.e., $W(\tilde{\rho})$ peaks sharply at $\tilde{\rho} = \exp(-3/2x)$ and then falls off in accordance with the logarithmically normal law upon moving away from this value. In the limit $x \to \infty$, only $\tilde{\rho} = 0$ and $\tilde{\rho} = \infty$ are attained. This should be expected to occur at $\eta = 0$, i.e., for absolutely exact levels.

The introduction of a finite η into (8) causes $W(\tilde{\rho})$ to fall off on the tails much more rapidly than in (9):

$$W\left(\tilde{\rho} \gg \frac{1}{8\eta}\right) \approx e^{-4\tilde{\rho}\eta}; \quad W(\tilde{\rho} \ll 8\eta) \sim e^{-4\eta/\tilde{\rho}}. \quad (10)$$

At $n \gg e^{-(x/2)}$ the function $W(\rho)$ no longer depends on x. In this case (7) should be substituted into (3) not only for L_m but also for R_m. The simplest expressions in this case are those for the state-density moments $\langle \rho^n\rangle$ and cumulants $\langle \rho^n\rangle_c$ at $\Delta = 0$:

$$\langle \rho_\eta^n\rangle = K_{n-{}^1/_2}(8\eta)/K_{-1/2}(8\eta); \quad \langle \rho_\eta^n\rangle_c = (4\eta)^{1-n}\frac{\Gamma\left(n-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}. \quad (11)$$

It follows from (11) that $W(\rho)$ has the form of an inverse Gaussian distribution

$$W(\rho) = \sqrt{\frac{4\eta}{\pi}}\frac{1}{\rho^{3/2}}\exp\left(-4\eta\frac{(\rho-1)^2}{\rho}\right). \quad (12)$$

5. To interpret the results obtained by us, we note that even if $\eta = 0$, in the presence of an open boundary the electronic levels have a finite width η_μ, which is associated with the departure of an electron from the system. In this case ρ will have the form as in (2), after the substitution of η_μ for η. Because of the localization of the electronic states in a 1D conductor, η_μ decreases exponentially with the distance from the boundary: $\eta_\mu \sim \exp(-2\alpha_\mu x)$, where α_μ is the reciprocal of the localization length of the μth level, which is centered near the point x. It is natural to assume that the

distribution of α_μ is in accordance with the normal law $W(a) \approx (x/8\pi)^{1/2}\exp[-x/8(\alpha-1)^2]$. This result, along with the fact that η_μ depends exponentially on α_μ, accounts for the logarithmically normal law (9). We see that the principal contribution to $\rho_{\eta,\Delta}$ comes from the level nearest in energy, which is centered near the point x. The fluctuations of ρ are determined primarily by the fluctuations in the width of this level, rather than by the distance from it.

To explain the logarithmically normal law (9), we need to know not the particular features of the 1D case, but rather only the fact that the electronic states are localized. For $\eta = 0$ and arbitrary dimensionality, the law like (9) should therefore be valid in the insulating state.

At $\eta \gtrsim \exp(-x/2)$ the overall decay of the levels ceases to fluctuate, and the behavior of $W(\rho)$ is determined by the spatial and energy positions of the levels. Consequently, the function $W(\rho)$ of the type in (10) and (12) is also a universal function for the insulating phase in a system of arbitrary dimensionality. In this case η can be specified, for example, by the temperature or inelastic relaxation.

6. It was shown in Ref. 4 that the nth moment of the complete state-density fluctuation, $\delta\nu$, is proportional to $\exp[n(n-1)\ln(\sigma_0/\sigma)]$, where σ_0 and σ are the classical and renormalized conductivities, respectively. This result is in excellent agreement with (9) if it is assumed that $\sigma \propto \sigma_0\exp(-x/8)$. On the other hand, our result seems to contradict Wegner's[8] assertion that $\ln\langle\rho^n\rangle$ increases with increasing n at a rate no greater than linear.

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