Theory of self-induced transparency in a focused light beam

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A theory is derived for the self-induced transparency of Gaussian light beams. A solution found in the single-mode approximation is a product of McCall–Hahn soliton and a Laguerre–Weber function. The carrier frequency of the pulse is redshifted from the transition line. The shift is inversely proportional to the radius of the minimum light spot.

In a well-known study, McCall and Hahn[1] observed a self-induced transparency during the coherent interaction of ultrashort intense light pulses with a medium. It was shown theoretically and experimentally in Refs. 2 and 3 that 2π pulses with a flat wavefront (the model of Ref. 1) are unstable with respect to transverse perturbations. Experiments by Egorov and Reutova[4] showed that a short, intense pulse with a converging beam shape propagates anomalously large distances with essentially no absorption in a resonant-absorption medium. The inadequacy of the plane-wave approximation has made it necessary to generalize the coherent interaction of short pulses with a medium with an inhomogeneously broadened line to the three-dimensional case.

The wave equation is analyzed in the radially symmetric case in the approximation of slowly varying phases and amplitudes. We are interested in a steady-state autowave solution. We introduce a dimensionless wave variable $u=(t-z/v)/\tau$, where τ and v are the length and propagation velocity of the pulse. We seek a solution of the wave equation in the form

$$\mathcal{E}(u,r)\exp(-i(\omega t-kz)).$$

To determine the slowly varying amplitude $\mathcal{E}(u,r)$ we have a parabolic equation in terms of the dimensionless coordinates $\rho_0=r/r_0$, $\sigma_0=k(c-v)/cv$, $k=\omega/c$, where c is the velocity of light in the medium:

$$\frac{\partial^2}{\partial\rho_0^2}\mathcal{E}+\frac{1}{\rho_0}\frac{\partial}{\partial\rho_0}\mathcal{E}-2i\sigma_0\frac{r_0^2}{\tau}\frac{\partial}{\partial u}\mathcal{E}=-4\pi k^2 r_0^2 P. \qquad (1)$$

We expand $\mathcal{E}$ and $\mathcal{P}$ in series in orthonormal Laguerre–Weber functions $\rho=\rho_0/\sqrt{1+(u-u_1)^2/u_0^2}$ $\tan\psi=u-u_1/u_0$,

$$D_n(u,r)=\frac{e^{i(2n+1)\psi}}{\sqrt{1+(u-u_1)^2/u_0^2}\sqrt{\pi}}\left(\frac{1}{n!}e^{\rho^2}\frac{d^n}{d(\rho^2)^n}e^{-\rho^2}\rho^{2n}\right)e^{-\rho_2^2(1+i\,\mathrm{tg}\,\psi)} \qquad (2)$$

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which constitute a solution of the parabolic equation. The dispersion of a Gaussian beam, r_0^2, and the parameter u_0 are related by

$$u_0 = \frac{\sigma_0 r_0^2}{\tau}. \tag{3}$$

We now write expansions for $\mathscr{E}$ and $\mathscr{P}$:

$$\mathcal{E} = \frac{\hbar}{d\tau}\sum_{n=0}^{\infty} E_n(u) D_n(u,r),$$

$$\mathcal{P} = \frac{\hbar}{d\tau}\sum_{n=0}^{\infty} P_n(u) D_n(u,r). \tag{4}$$

Making use of the orthogonality of the Laguerre polynomials, we find a relationship between the coefficients from Eq. (1):

$$\frac{d}{du}E_n(u) = -i\frac{2\pi k^2\tau}{\sigma_0}P_n(u). \tag{5}$$

Since the pulse is short, we can ignore relaxation processes in the medium. The Block equations for the medium become

$$\frac{d\mathcal{P}}{du} - i\tau\Delta\omega\mathcal{P} = \frac{id\tau}{\hbar}N\mathcal{E},\quad \frac{dN}{du} = -i\frac{\tau}{2\hbar}(\mathcal{E}\mathcal{P}^* - \mathcal{E}^*\mathcal{P}), \tag{6}$$

where $\Delta\omega = \omega - \omega_{ab}$. Eliminating the population N from Eqs. (6), we find a linear second-order differential equation for the polarization $\mathscr{P}$. Substituting expansion (4) into it, and using (5), we find an infinite system of coupled nonlinear third-order differential equations for the coefficients $E_n(u)$. We are interested in separate single-mode equations, in accordance with the experiments of Ref. 4:

$$E_n\frac{d^3}{du^3}E_n + \left[E_n\frac{d^2}{du^2}E_n - \left(\frac{d}{du}E_n\right)^2\right] i(a_n - \Delta\omega\tau)$$

$$-\frac{d}{du}E_n\frac{d^2}{du^2}E_n + \frac{\mu_n}{2}E_n^2\frac{d}{du}|E_n|^2 = 0, \tag{7}$$

where

$$\mu_n = \int |D_n|^4 r dr d\varphi,\quad a_n = \frac{2n+1}{2u_0}. \tag{8}$$

Equation (7) is written in first order in a perturbation theory in the small parameter u_0^{-1}. Using (1)–(8), we find an autowave solution for the complex field amplitude:

$$\mathcal{E}_n(u) = \frac{\hbar}{d\tau}\frac{2}{\sqrt{\mu_n}}\operatorname{sech} u D_n(u,r)\exp\left\{-i\left(\frac{2n+1}{4u_0} - \frac{\omega-\omega_{ab}}{2}\tau\right)u\right\}. \tag{9}$$

It follows from an analysis of solution (9) that the relation

$\sqrt{\mu_n}\max|\mathscr{E}_n| = \text{const}$ holds for the field amplitude, with $\sqrt{\mu_0} = 1/\sqrt{2\pi} > \sqrt{\mu_1} = 1/\sqrt{4\pi} > ...$. The mode amplitude thus increases with the mode index. There exists a threshold pulse intensity, above which there is an autowave solution for only the zeroth mode. The thresholds for the other modes are correspondingly higher; this circumstance is helpful for singling out a single-mode solution. In the experiments of Ref. 4, an additional focusing of a single-mode laser beam by a lens was of assistance in establishing an autowave solution of the type in (9). This focusing was carried out in such a way that the focus was directly behind the cell holding the absorbing medium. A pulse with a converging geometry of a Gaussian beam propagated through the medium.

For single-mode solution (9), the energy conservation law[5]

$$c\frac{\partial}{\partial z}\frac{|E_n(t-z/v)|^2}{4\pi} + \frac{\partial}{\partial t}\frac{|E_n(t-z/v)|^2}{4\pi} + \hbar\omega\frac{\partial N_n}{\partial t} = 0 \tag{10}$$

holds. By virtue of this conservation law, the medium and the field exchange energy, without loss. Conservation law (10) determines the pulse propagation velocity:

$$v^{-1} = c^{-1}\left(1 + \frac{\omega\tau^2}{T_n}\right), \tag{11}$$

where

$$T_n = \frac{\hbar}{2\pi\mu_n d^2 n_0}. \tag{12}$$

This velocity is much higher than in the one-dimensional case, since we have $\mu_n \ll 1$. The velocity increases with the mode index.

The Gaussian structure of the beam results in a frequency shift, which cannot be found for a plane-wave solution. The steady-state propagation of the autowave solution is characterized by the shifted frequency

$$\omega = \omega_{ab} - \frac{3}{2}\frac{(2n+1)}{u_0\tau}. \tag{13}$$

The shift is in the red direction from the transition line, in agreement with the experiments of Ref. 4. Using expressions (3), (11), and (12), we can link the shift $(u_0\tau)^{-1}$ with quantities which can be measured experimentally: the minimum radius of the light spot (r_0), the pulse length τ, and the density n_0. This relationship is

$$(\tau u_0)^{-1} = \frac{T_n}{(kr_0)^2\tau^2}. \tag{14}$$

When we go over to a plane wave, $(kr_0)^{-2}\to 0$, the Gaussian beam spreads out, the frequency shift vanishes, and we obtain the McCall–Hahn solution. A significant frequency shift in the red direction was noted in Ref. 4. By virtue of that shift, even a pulse which was not very short ($T_2/\tau \approx 3$) propagated anomalously large distances ($k_0 L \approx 60$).

A significant frequency shift was not observed in earlier experiments. The probable reason is the strong dependence of the shift on the radius of the Gaussian beam, $(u_0\tau)^{-1} \sim r_0^{-2}$. The use of a converging beam geometry makes it possible to bring out this effect.

To pursue the analysis, we rewrite expression (14) in the form

$$(\tau u_0)^{-1} = \frac{\omega}{2(kr_0)^2} S, \quad S = \frac{max|E_n|^2/4\pi}{\hbar\omega n_0}. \tag{15}$$

We see from this expression that the frequency shift is proportional to the energy of the electromagnetic radiation field divided by the maximum energy which the medium is capable of absorbing. In order to satisfy the condition $(\tau u_0)^{-1} = \text{const}$, we need $S = \text{const}$; this condition is in qualitative agreement with the experiments of Ref. 4. By virtue of our perturbation theory in the parameter u_0^{-1}, expression (15) becomes inapplicable at very large values of $\max|E_n|^2$, or at small values of N_0. This case, $u_0 \ll 1$, has not previously been studied theoretically, but the experiments of Ref. 4 suggest that (15) should be generalized to the case $S \gg 1$. A saturation apparently arises in terms of the parameter S. We can thus argue that the empirical formula

$$(\tau u_0)^{-1} = \frac{\omega}{2(kr_0)^2} \frac{S}{1+S^2} \tag{16}$$

is valid.

From this discussion we can draw the following conclusions. (1) The autowave solution found here describes a self-induced transparency accompanying the propagation of a pulse of a Gaussian light beam through a resonant medium with a homogeneously broadened spectral line. (2) In agreement with the experiments of Ref. 4, the solution contains a characteristic frequency shift of the pulse, in the red direction from the center of the absorption line. A spatially focused and temporally compressed light pulse is realized if the refractive index increases in this space-time region. This increase can be arranged through the use of an appropriate spatial and frequency dispersion. Such a dispersion occurs in a Gaussian beam. Because of the self-consistent nature of the pulse propagation, the frequency dispersion puts the carrier frequency of the pulse on the long-wavelength wing of the absorption line. This circumstance in turn promotes, through the dispersion mechanism, an increase in the refractive index in the region occupied by the pulse. The magnitude of the shift is inversely proportional to the square of the minimum radius of the light spot and to the square of the pulse length, (14). (3) The pulse propagation velocity is much higher than that for a plane wave, and it increases with increasing index of the transverse mode [see (11)]. (4) The solution found here can be generalized to the case of a resonant medium with an inhomogeneously broadened absorption line.

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