# Relationship between exactly solvable and quasiexactly solvable versions of quantum mechanics with conformal-block equations in 2D theories

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A relationship is found between the semiclassical limit of the conformal-block equations in a 2D conformal theory and special 1D versions of quantum mechanics.

Morozov *et al.*[1] have suggested an analogy between rational 2D conformal field theories and the "quasiexactly solvable" problems of quantum mechanics which were recently discovered.[2-4] Morozov *et al.* hoped that it would be possible to find some entities in quantum mechanics which have the meaning of conformal blocks, structure

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constants of an operator algebra, and a spectrum of conformal dimensionalities. In the present letter we derive exact relations between (on the one hand) exactly solvable problems and quasiexactly solvable problems and (on the other) three- and four-point conformal blocks in a theory with a zero vector at the second level.

Let us review some information about exactly solvable and quasiexactly solvable problems which we will need below. These cases are distinguished by the circumstance that they can be described in a natural way in terms of the $SL(2,R)$ group.[2–5] Specifically, the Hamiltonians of special versions of quantum mechanics are of the form

$$H = C_{ab} J^a J^b + C_a J^a, \tag{1}$$

where $C_{ab}$ and $C_a$ are constants, while the $J^a$ are the generators of $SL(2,R)$ in the differential representation:

$$J_0 = -j + \xi \frac{d}{d\xi}, \quad J_+ = 2j\xi - \xi^2 \frac{d}{d\xi}, \quad J_- = \frac{d}{d\xi}. \tag{2}$$

Here $\xi$ is the variable of a group manifold, and $j$ is the spin of the representation. In differential form, Hamiltonian (1) becomes

$$H = Q_4(\xi) \frac{d^2}{d\xi^2} + Q_3(\xi) \frac{d}{d\xi} + Q_2(\xi), \tag{3}$$

where $Q_n(\xi)$ is a polynomial of degree $n$ in $\xi$. After we carry out the necessary diffeomorphisms and gauge transformations, Hamiltonian (3) takes the standard form

$$H = -\frac{\partial^2}{\partial x^2} + V(x), \quad x = \int^{\xi} d\xi' / \sqrt{Q_4(\xi')}. \tag{4}$$

The condition under which this problem is exactly solvable reduces to the condition that (3) must be independent of the spin of the representation, $j$. Then $H_{\mathrm{ex}}$ reduces to

$$H_{\mathrm{cs}} = Q_2(\xi) \frac{d^2}{d\xi^2} + Q_1(\xi) \frac{d}{d\xi}. \tag{5}$$

The number of fixed points in the case of diffeomorphism (4) is two for exactly solvable problems and three or four for quasiexactly solvable problems. For the discussion below, it is convenient to use another description of the versions of quantum mechanics, which was proposed in Ref. 3. For this purpose, we rewrite the spectral problem $H\Psi = E\Psi$ in the form

$$Q_4(\xi) \left[ \frac{d^2}{d\xi^2} - \sum_{\alpha=1}^{4} \frac{b_\alpha}{\xi - \xi_\alpha} \frac{d}{d\xi} + \sum_{\alpha=1}^{4} \frac{C_\alpha}{\xi - \xi_\alpha} \right] \Psi(\xi) = 0, \tag{6}$$

where the spectral parameter is included in $C_\alpha$. The problem of determining the spectrum reduces to one of determining the $c$ numbers at fixed values of $b_\alpha$ and $\xi_\alpha$. After a gauge transformation, Eq (6) takes the canonical form

$$\left[\frac{d^2}{d\xi^2}-\sum_\alpha \frac{b_\alpha(b_\alpha-1)}{(\xi-\xi_\alpha)^2}+\sum_\alpha \frac{C_\alpha}{\xi-\xi_\alpha}\right]\tilde{\Psi}=0. \tag{7}$$

As was shown in Ref. 3, the $C_\alpha$ represent the spins of the representations of $SL(2,R)$, which specify the potential. For exactly solvable problems we have $\alpha = 1,2$; for quasiexactly solvable problems we have $\alpha = 1$–$3$ or $1$–$4$. The spins obey the constraint

$$\sum_\alpha b_\alpha = 1-2j. \tag{8}$$

Ushveridze[3] has also proposed a useful interpretation of the problem of finding the spectrum: as the problem of finding the equilibrium state of mobile charged particles (the zeros of wave functions) in a field of immobile dyons at the points $\xi_\alpha$.

We turn now to a description of the conformal blocks in a 2D theory with a zero vector in the second level. The condition for the splitting of the zero vector makes it possible to find a conformal-block equation in the semiclassical limit $c\to\infty$, where $c$ is the central charge:

$$\left(\frac{d^2}{dz^2}+\frac{1}{4}\sum_{i=1}^{N-1}\frac{1-m_i^2}{(z-z_i)^2}+\frac{1}{2}\sum_{i=1}^{N-1}\frac{c_i}{z-z_i}\right)y_c(z)=0. \tag{9}$$

The conformal blocks are related to $y_c(z)$ by

$$\int_c dt < J_+(t)V_{2,1}(z)\prod_{i=1}^{N}V_i(z_i) >= \prod_{i<j}(z_i-z_j)^{-\frac{(1-m_i)(1-m_j)}{\gamma^2}}y_c(z), \tag{10}$$

where $J_+(t)$ is a screening operator, $V_{2,1}$ is the null vector, the $V_i(z)$ are vertex operators with classical conformal dimensionalities $\Delta_i = \frac{1}{2}(1-m_i^2)$, and $\gamma^2 = 2/3c^2$. The neutrality condition leads to the relation

$$\sum_{i=1}^{N}(1-m_i)=2. \tag{11}$$

The topology of a sphere is to be understood in (9).

Comparing Eqs. (7) and (9), we can make the identification

$$b_\alpha=\frac{1-m_\alpha}{2},\ \ \Delta_\alpha=-2b_\alpha(b_\alpha-1). \tag{12}$$

The classical conformal dimensionality is therefore expressed in terms of the spin of the $SL(2,R)$ representation. Since the number of poles of the potential in (7) is usually $K = 2$ for exactly solvable problems and $K = 3$ for quasiexactly solvable problems, we can assert that there is a relationship between exactly solvable problems and three-point entities, and there is also a relationship between quasiexactly solvable problems and four-point entities, in the conformal theory. The relationship between the conformal dimensionalities and the eigenvalues of the operator $J^2$ corresponds to a Sugawara construction for the Kac–Moody $SL(2,R)$.

The analogy between (7) and (9) explains the meaning of an operator expansion in quantum mechanics. An operator expansion of the type

$$V_{\Delta_1}(z_1)V_{\Delta_2}(z_2) \to \frac{C^{\Delta_3}_{\Delta_1\Delta_2}}{(z_1 - z_2)^{\Delta_3-\Delta_1-\Delta_2}} V_{\Delta_3}(z_1) \tag{13}$$

can be reformulated in terms of a relationship between the wave functions of quasiexactly solvable and exactly solvable problems. From the standpoint of quantum mechanics, a change in the number of vertex operators in (13) corresponds to a decrease in the number of poles in potential (7) by one, and the analog of (13) is

$$\Psi^{\rm qes}_{b_1,b_2}(z_1,z_2) = \sum_b \frac{C^b_{b_1 b_2}}{(z_1 - z_2)^\alpha} \Psi^{\rm es}_b(z_1). \tag{14}$$

In other words, the structure constants $C^b_{b_1 b_2}$ correspond to the coefficients of an expansion of the wave functions of the quasiexactly solvable problem in terms of the wave functions of the exactly solvable problem. From the standpoint of quantum mechanics, the operator expansion is formulated in the parameter space of the problem, so we would naturally expect singularities which would lead to nontrivial Berry phases (Ref. 7). In our case, the parameter space is three-dimensional, so monopole singularities arise at the points at which the levels of the Hamiltonian cross, as functions of $(\xi_1,\xi_2,\xi_3)$. In the picture of Coulomb charges, the structure of expansion (14) corresponds to different types of merging of charges and to the formation of monopoles. We might also note that our point of view regarding the operator expansion in quantum mechanics disagrees with the hypothesis offered by Morozov *et al.*[1] — that the analog of this expansion is a relationship which is trilinear in the wave functions.

Gomez and Sierra[6] have pointed out a relationship between semiclassical equation (9) and an equation which determines the "uniformization" of a Riemann surface with the points listed above. In this case, (11) plays the role of the Riemann–Roch relation for 2-differentials on a Riemann sphere. The situation corresponds to $j = 0$ in quantum mechanics. A generalization to higher types leads to (8) with arbitrary $j$, so one would naturally assume that Eq. (7) has an arbitrary value of $j$. One can then algebraically determine the conformal blocks for arbitrary $j$ for the exactly solvable problems and thus for three-point entities, while for four-point entities one can do this up to $j$.

The picture sketched here will be analyzed in detail in a separate publication.

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[1] A. Yu. Morozov *et al.*, Int. J. Mod. Phys. A **5**, 803 (1990).
[2] A. V. Turbiner, Commun. Math. Phys. **118**, 467 (1989).
[3] A. V. Ushveridze, Fiz. Elem. Chastits At. Yadra **20**, 1185 (1989) [Sov. J. Part. Nucl. **20**, 504 (1989)].
[4] M. A. Shifman, Int. J. Mod. Phys. A **4**, 2897 (1989).
[5] Y. Alhassid, F. Gursey and F. Iachello, Ann. Phys. (NY) **148**, 346 (1983).
[6] C. Gomez and G. Sierra, Phys. Lett. **255**, 51 (1991).
[7] M. V. Berry, Proc. R. Soc. (London) **45**, 392 (1984).

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