Marginal deformations of conformal field theories with Z_3 symmetry

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Nontrivial deformations of a W_3 Zamolodchikov algebra which conserve the central charge are found.

Progress has recently been made toward solving the problem of classifying conformal field theories. This problem is part of string theory. One of the most interesting of the various directions that are developing involves the W algebras which were discovered (for several special cases) by Zamolodchikov.[1] Substantial progress has been made toward a systematic classification of W algebras on the basis of the BBSS construction.[2] The recent advances were put in systematic form in Ref. 3, where a powerful method was proposed for constructing W algebras on the basis of the Jacobi identities. Many questions remain open, however. Some were discussed in a paper by Morozov.[4] In particular, there is the question of the existence of continuous deformations of W algebras which conserve the central charge c. Our purpose in the present study was to solve this problem. Below we present an explicit construction of a family of W algebras which allow marginal deformations.

Continuous deformations of conformal theories which conserve the central charge c were found by Morosov *et al.*[5,6] for an energy-momentum tensor written in the quadratic form

$$T = t_{ab} \,{}^*_*J_a J_b{}^*_* \tag{1}$$

on the current algebra $SL(2)_4$. They derived the solution by means of a construction which they developed independently and which had also been developed slightly earlier by Halpern and Kiritsis.[7] Several authors have subsequently found many different chiral conformation theories which allow marginal deformations. In particular, we showed in Ref. 8 that the space of deformations of conformal theories of the type in (1) with a diagonal inertial tensor t_{ab} and a current algebra $SO(N)_2$ is a Grassmann manifold $G_c(R^{N-1})$. Halpern and Obers[9] made a similar prediction independently, on the basis of slightly different considerations, although they did not present an explicit construction.

We turn now to a direct description of our construction. We restrict the discussion to the case of the W_3 algebra for simplicity, but corresponding constructions could be carried out for other W algebras. The W_3 algebra is formed by two fields, T and Q, of spins 2 and 3, respectively. The operator expansion for $Q(z)$ and $Q(w)$ is[1]

$$Q(z)Q(w) = \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3}$$

$$+\frac{1}{(z-w)^2}\left\{2b^2\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right\} + O\left(\frac{1}{z-w}\right), \tag{2}$$

where $A(z) = {}^\times_\times TT {}^\times_\times (z) - (3/10)\partial^2 T(z)$ and $b^2 = 16/22 + 5c$. We seek T and Q as differential polynomials on the algebra of currents $U(1)^N_k$:

$$\begin{aligned} T &= t_{ij} {}^*_* H_i H_j {}^*_* \\ Q &= M + G, \\ M &= m_{ijk} {}^*_* H_i H_j H_k {}^*_* \\ G &= g_{ij} {}^\times_\times H_j \partial H_j {}^\times_\times, \end{aligned} \tag{3}$$

where ${}^{**}_{**}$ is the symmetrized normal product, and ${}^{\times\times}_{\times\times}$ is the ordinary normal product.

A generalization of the construction of an affine-Virasoro control equation[5,7] to the case of W algebras results in a system of equations for the structure constants of the theory: t_{ij}, m_{ijk}, and $g_{ij} = -g_{ji}$. Substituting in the ansatz

$$\begin{aligned} t_{ij} &= \frac{1}{2k} R_i^\mu R_j^\mu, \\ m_{ijk} &= \frac{1}{3k^{3/2}} R_i^\mu R_j^\nu R_k^\lambda \mu_{\mu\nu\lambda}, \\ g_{ij} &= \frac{1}{k} R_i^\mu R_j^\nu \gamma_{\mu\nu}, \end{aligned} \tag{4}$$

where the R_i^μ form an orthogonal c-hedron in R^N,

$$R_i^\mu R_i^\nu = \delta^{\mu\nu}, \tag{5}$$

we find a system of equations for the $O(c)$ tensors $\mu_{\mu\nu\lambda}$ and $\gamma_{\mu\nu}$:

$$\begin{aligned} \mu_{\mu(\kappa\rho}\mu_{\sigma\tau)\mu} &= \frac{b^2}{2}\delta_{(\kappa\rho}\delta_{\sigma\tau)}, \\ \mu_{\mu\mu\lambda} &= 0, \\ \gamma_{\mu\nu}\gamma_{\mu\lambda} &= \frac{1}{10}(1-2b^2)\delta_{\nu\lambda}. \end{aligned} \tag{6}$$

Solutions of these equations exist for $c = 2$ and $c = 8$. The corresponding deformation spaces are the equivalence classes of the c-reference basis of the relative action $O(c)$, i.e., the Grassmann manifold $G_c(R^N)$.

We have thus demonstrated the existence of deformations of W_3 algebras which conserve central charge.

Using the vertex-operator construction

$$E_\alpha = {}^\times_\times e^{i\alpha\cdot\varphi} {}^\times_\times c_\alpha,$$
$$H_i = i\partial\varphi_i, \tag{7}$$

where the cocycles c_α satisfy $c_\alpha c_\beta = \epsilon(\alpha\beta) c_{\alpha+\beta}$, $\epsilon(\alpha\beta) = \pm 1$, we can perform an immersion of the deformation space of W_3 algebras of the type in (3) with $k=1$ into the deformation space of W_3 algebras constructed in the form of differential polynomials on the algebra $SU(N)_1$:

$$T = \sum_\alpha T_\alpha {}^\times_\times E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha {}^\times_\times,$$
$$Q = \frac{1}{6} \sum_{\alpha\beta\gamma}' S_{\alpha\beta\gamma} {}^*_* E_\alpha E_\beta E_\gamma {}^*_* + \frac{1}{6} \sum_{\alpha\beta\gamma}' A_{\alpha\beta\gamma} {}^*_* E_\alpha E_\beta E_\gamma {}^*_*$$
$$+ \sum_{\alpha\in\Delta_+} U_\alpha {}^\times_\times \nu_1 (E_\alpha \partial E_{-\alpha} - E_{-\alpha} \partial E_\alpha) + \nu_2 [E_\alpha \partial E_{-\alpha}] {}^\times_\times, \tag{8}$$

where Σ' means $\alpha + \beta + \gamma = 0$, Δ_+ is the system of positive roots, $S_{\alpha\beta\gamma} = S_{-\alpha,-\beta,-\gamma}$ and $A_{\alpha\beta\gamma} = -A_{-\alpha,-\beta-\gamma}$.

For this purpose it is sufficient to substitute the following equations into ansatz (8):

$${}^\times_\times E_\alpha E_{-\alpha} + E_{-\alpha} E_\alpha {}^\times_\times = {}^\times_\times (\alpha \cdot \vec{H})^2 {}^\times_\times,$$
$${}^\times_\times E_\gamma {}^*_* E_\alpha E_\beta {}^*_* + E_{-\gamma} {}^*_* E_{-\alpha} E_{-\beta} {}^*_* {}^\times_\times = \frac{1}{2} \epsilon(\alpha\beta) {}^\times_\times (\alpha - \beta, \vec{H})(\alpha + \beta, \vec{H})^2 {}^\times_\times,$$
$${}^*_* E_\alpha E_\beta E_\gamma - E_{-\alpha} E_{-\beta} E_{-\gamma} {}^*_* = -\frac{1}{2} \epsilon(\alpha\beta) {}^\times_\times (\alpha \cdot \vec{H})(\beta \cdot \partial\vec{H}) - (\beta \cdot \vec{H})(\alpha \cdot \partial\vec{H}) {}^\times_\times. \tag{9}$$

These equations can be found from the operator expansion for the fields $E_\alpha(z)$ through the use of substitution (7).

We have thus also succeeded in constructing nonAbelian realizations of deformable W algebras. It would be interesting to generalize the construction presented here to the case of other W algebras.

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