Collapse of a dusty medium and Titius–Bode law in natural units

A. É. Filippov
Physicotechnical Institute, Academy of Sciences of the Ukrainian SSR, 340114, Donetsk

(Submitted 22 August 1991)
Pis'ma Zh. Eksp. Teor. Fiz. **54**, No. 7, 357–360 (10 October 1991)

If a protoplanetary cloud forms as a pancake of collapsing dust, the system acquires a special length scale: the orbit of the most massive planet. This scale determines the sequence of the orbits of other planets. In a zeroth approximation, these other orbits satisfy a simple rule involving resonances of orbital periods.

A likely scenario for the formation of a protoplanetary disk would be the onset of an instability in the motion of a dusty medium, with different regions moving at different velocities. This process would be qualitatively similar to the process by which galaxies form.[1] A necessary condition here is that a volume element of the medium shrink along some direction, forming a fold catastrophe for the density function. It can be shown that the most likely result of such a process would be the formation of a flat ellipsoid or "pancake."[1,2] The compression and heating of the dust at the surface of this pancake would stop the motion of the particles in the direction normal to this surface, while the tangential components of the velocities of oppositely directed dust streams colliding at the surface of the pancake would essentially be conserved. Since the flow velocities are generally not parallel even in the case of a potential flow, this process should lead to the appearance of a torque.[3,4] Vorticity nonconservation would be possible because entropy would increase in the plane of the pancake.[1]

As a result of this process, the pancake would have a certain tangential velocity component v_0. The existence of this special value v_0 should have observable consequences in the structure of the planetary system. The reason is that, if all the particles in the pancake had an identical velocity v_0, then in the gravitational field of the pancake (basically, in the field of its central bulge) these particles would move along trajectories which would allow the formation of a single planet. In particular, in the approximation of a circular orbit the fixed value v_0 would determine a radius $r_0 \propto 1/v_0^2$. Actually, however, the particle velocities are distributed in some random way around v_0. This velocity dispersion means that particles can orbit at an arbitrary radius $r \neq r_0$. Nevertheless, if a special velocity v_0 does exist, the r profile of the dust will be definitely nonuniform even before it begins to break up into collapsing rings.[1,5] At the outset we might expect the formation of a planet with most of the mass of the system near $r = r_0$.

Let us estimate the density $\rho(r)$ of the flow of particles orbiting at a radius r. We assume a normal distribution of the velocities around v_0 (in the case at hand, the normal distribution can be assumed to be in a steady state, in contrast with that at the scale of galaxies[1]): $f(v) \propto \exp[-(v-v_0)^2/\bar{\sigma}]$. The density $w(r;b)$ of the pancake which forms in the course of the fold catastrophe also depends on r, falling off with the radius (with some standard deviation b). In a cylindrically symmetric system with

$v(r) \propto 1/r^{1/2}$ we would have the estimate

$$\rho(r) \propto rw(r;b)\exp[-(1-1/r^{1/2})/\sigma]. \tag{1}$$

The special nature of r_0 qualifies it as a natural length scale of the system, so we have $r_0 = 1$ in (1). It is also natural to choose $\rho(r=1)=1$. In these units, $\rho(r)$ becomes a universal quantity. Its structure is determined exclusively by the standard deviations of the functions f and w. Figure 1 shows the function $\rho(r)$ for $\sigma = 0.6$ and $b = 2$. This function has characteristic features which qualify it as a good zeroth approximation for the mass distribution in the real solar system (correspondingly, in units of the Jovian mass). The most important point is that this function determines the sharp difference between the masses of the inner and outer planets from the standpoint of Jupiter, by virtue of the factor $\propto \exp[-(1-1/r^{1/2})/\sigma]$. We might also note the good agreement between $\rho(r)$ and the masses of the planets in the extreme positions. This approximation is at its worst for the planets near Jupiter. The reason is the redistribution of the mass of the protoplanetary disk, with a shift of mass toward the region of greatest condensation, at $r \simeq r_0$ (Ref. 1).

The fairly good agreement between the seed distribution $\rho(r)$ and the masses $m(r_k)(k=1,2,...)$ is evidence that a special scale does indeed exist in this system, even in an early stage of its evolution. It is natural to use this scale to analyze the Titius-Bode law, which specifies the positions of planetary orbits (more precisely, to analyze a corresponding law expressed in units of r_0).

The Titius–Bode law has historically been written in the form

$$\bar{r}_k = 0,1\cdot[3\cdot 2^k + 4] \equiv \alpha\beta^k + \gamma, \tag{2}$$

where $\bar{r}_k$ is expressed in astronomical units, and we have $k \to -\infty$ for Mercury, $k=0$ for Venus, and $k = 1,2,...$ for the other planets. Because of the arbitrary choice of the astronomical unit as the length scale, we are obliged to use a greater number of adjus-

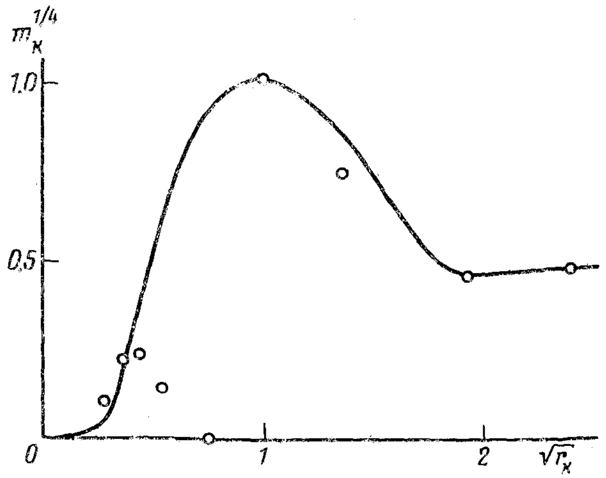

FIG. 1. The profile $\rho(r)$ in the case in which there is a special length scale r_0. The circles represent the masses of the planets [with $m(r_0) = m_{k=6} = 1$].

table parameters, α, β, and γ (and the limit $k \to -\infty$ is of course quite contrived). For this reason, Eq. (2) is difficult to comprehend.

No matter how accurate these arguments about the appearance of a length scale r_0 may be, it is quite obvious that the most massive planet (i.e., Jupiter, which has two-thirds of the mass of the entire system) plays a special role in coordinating the motion of the overall system. The quantity r_0 itself is a natural unit for a law like (2). A direct calculation shows that

$$r_k = (2.5)^{\frac{2}{3}(k-6)} \tag{3}$$

is a fairly good approximation. Alternatively, using Kepler's third law, we find the following simple relation for the orbital periods T_k, with $T_6 = 1$:

$$T_k = (2.5)^{(k-6)}. \tag{3a}$$

In other words, this "idealized" motion of the system forms a chain of resonances such that the orbital period of each planet in the series is 2.5 times the period of the preceding planet, and the fundamental mode of this motion is specified by the most massive of the planets.[1)] It can be seen from Fig. 2 that relation (3a) holds well for the planets closest to Jupiter (we are using an average orbit of the asteroids). These are the planets which suffered the greatest loss of mass to Jupiter (Fig. 1). As a result, the planets coming after these planets (starting with Jupiter) exhibit a sort of tendency to

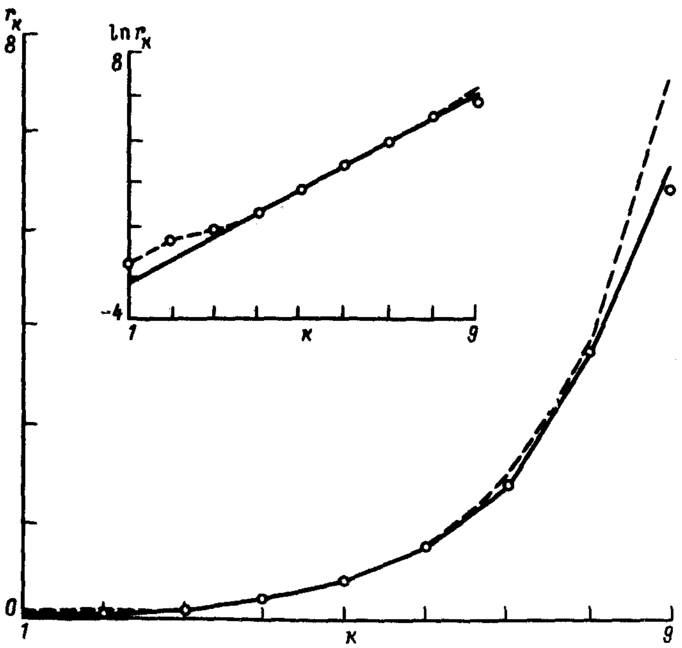

FIG. 2. A fit of the function $r = (2.5)^{\frac{2}{3}(k-6)}$ (the solid line) to the sequence r_k (the circles). The dashed line is the Titius–Bode function. The inset shows the same results in logarithmic scale.

deviate from the relation $T_{k+1}/T_k = 2.5$ in favor of other resonances. Interestingly, for T_2/T_1 the ratio $T_2/T_1 = 2.5$ arises again, so the greatest deviation from relation (3a) is found for the orbit of the Earth, with $k = 3$. As a result, the use of the astronomical unit as the length scale is most unfortunate. It greatly obscures the nature of the r_k sequence.

[1)] The particular value $T_1/T_k = 2.5$ is probably determined by the standard deviation σ of the mass distribution. A fixed value of the ratio $r_{k+1}/r_k = r_k/r_{k-1}$, on the other hand, can be understood on the basis of the following qualitative arguments. Those layers of the pancake which are revolving in orbits with an instantaneous radius $r > r_k$ $(r < r_k)$ fall further behind (overtake) the layer $r = r_k$ at a relative velocity $\Delta v = |v(r) - v_k|$. The gravitational collapse of the pancake gives rise to condensing regions whose width is determined in order of magnitude by the condition $[v(r) - v_k]^2 = A/|r - r_k|$, where A is a form factor which depends on the standard deviation. In units of $y = r/r_k$ we have $z(y) = (y^{1/2} - 1)^2|y - 1|/y = A$. The solution of the equation $z(y) = A$ yields an estimate of the boundaries at which the matter of the pancake contracts toward the orbit of radius r_k and thus an estimate of the distances between adjacent orbits r_{k+1} and r_k. It is easy to verify that we have $z(y) = z(1/y)$ and thus $r_{k+1}/r_k = r_k/r_{k-1}$.

[1] Ya. B. Zel'dovich and I. D. Novikov, *The Structure and Evolution of the Universe*, U. Chicago P., Chicago, 1983.
[2] B. A. Trubnikov, Pis'ma Zh. Eksp. Teor. Fiz. **47**, 365 (1988) [JETP Lett. **47**, 437 (1988)].
[3] A. D. Chernin, Pis'ma Zh. Eksp. Teor. Fiz. **11**, 317 (1970) [Sov. Phys. Usp. **11**, 210 (1970)].
[4] A. G. Doroshkevich, Astrophys. Lett. **14**, 11 (1973).
[5] V. L. Polyachenko and A. M. Fridman, Zh. Eksp. Teor. Fiz. **94**(5), 1 (1988) [Sov. Phys. JETP **67**, 867 (1988)].

Translated by D. Parsons