Spinor Cartan moving n-hedron, Lorentz-harmonic formulations of superstrings, and κ symmetry

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Twistor-like formulations of $D = 10, N = 2B$, and $D = 4, N = 1$ superstrings are constructed through a harmonic generalization of the Newman–Penrose dyad formalism. The problem of constructing covariant irreducible generators of κ symmetry is automatically resolved in these formulations.

Covariant quantization is a key problem in superstring theory.[1] This problem was recently solved for null-super-p-branes ($p = 0$ for a massless superparticle; $p = 1$ for a null superstring; and $p = 2$ for a null supermembrane) in a $4D$ space-time through the use of a new Lorentz-harmonic formulation of their action.[2,3] A crucial aspect of the approach of Refs. 2 and 3 is the use of the Newman–Penrose dyads[4] T_α, O_α or their equivalent Lorentz harmonics[5] v_α^-, v_α^+ as auxiliary variables to supplement the coordinates of the null-super-p-branes. Lorentz harmonics are a spinor realization of a Cartan moving n-hedron. A projection of the Grassman spinor constraints onto these harmonics singles out the covariant and irreducible generators of κ symmetry. We thus regard it as conceptually important to use the variables of the spinor n-hedron to formulate the action of Green–Schwarz superstrings.

We will also present some formulations with a twistor-like action for $D = 10$, $N = 2B$ and $D = 4$, $N = 1$ superstrings, through a generalization of the action of null superstrings.[2] The generalization consists of incorporating a nonzero tension and an expansion of the representation of Ref. 2 to the $D = 10$ case.

Actions for $D = 10$, $N = 2B$ and $D = 4$, $N = 1$ superstrings can be written in a consistent manner by using light-like vectors[1)] $u_m^{(\pm 2)}$ constructed from the corresponding spinor Lorentz-harmonic variables (see Ref. 5 for $D = 4$ and Ref. 6 for $D = 10$):

$$S_{D,N} \equiv \int d\tau d\sigma \mathcal{L}(\tau,\sigma) = S_{1,D,N} + S_{WZW,D,N}, \tag{1}$$

$$S_{1,D,N} = -\frac{i}{2}\int d\tau d\sigma [\rho^{(+2)\mu}\omega_\mu^m u_m^{(-2)} + \varphi^{(-2)\mu}\omega_\mu^m u_m^{(+2)} + \alpha' \epsilon_{\mu\nu}\rho^{(+2)\mu}\varphi^{(-2)\nu}].$$

Here $\mu = 0,\ 1$ is the vector index of the world sheet; $\omega_\mu^m \equiv \partial_\mu x^m - i(\partial_\mu \theta^\alpha \sigma_{m\alpha\dot\alpha} \bar\theta^{\dot\alpha} - \theta^\alpha \sigma_{m\alpha\dot\alpha}\partial_\mu \bar\theta^{\dot\alpha})$ for $D = 4$, $N = 1$ ($m = 0,\ 1,\ 2,\ 3; \alpha = 1,2;$ $\dot\alpha = 1,\ 2; \sigma_{m\gamma\dot\gamma}$ are relativistic Pauli matrices) and $\omega_\mu^m \equiv \partial_\mu X^m - i(\partial_\mu \Theta^{\alpha 1}\sigma_{m\alpha\beta}\Theta^{\beta 1} + \partial_\mu \Theta^{\alpha 2}\sigma_{m\alpha\beta}\Theta^{\beta 2}$ for $D = 10$, $N = 2B$ ($m = 0, ..., 9; \alpha = 1, ...,$ 16; $\sigma_{m\alpha\beta}$ are ten-dimensional σ matrices).[7] We denote the coordinates of the ordinary

$D=10$, $N=2B$ and $D=4$, $N=1$ superspaces by $z^M=[x^m,\theta^\alpha,\bar\theta^\alpha\equiv\overline{(\theta^\alpha)}]$ and $z^M=(X^m,\Theta^{\alpha 1},\Theta^{\alpha 2}\equiv(X^m,\Theta^{\alpha 1})$, respectively.

The vectors $u_m^{(\pm 2)}$, which satisfy the light-like conditions $u_m^{(-2)}u^{(-2)m}=0=u_m^{(+2)}u^{(+2)m}$ and the orthonormality conditions $u_M^{(-2)}u^{(+2)m}=2$, are constructed from corresponding spinor Lorentz harmonics[5,6] in the following way:

$$D=4:u^{(-2)m}\equiv u^{(+|-)m}=v^{\alpha-}\sigma^m_{\alpha\dot\alpha}\bar v^{\dot\alpha+},$$

$$u^{(+2)m}\equiv u^{(-|+)m}=v^{\alpha+}\sigma^m_{\alpha\dot\alpha}\bar v^{\dot\alpha-}, \tag{2a}$$

$$D=10:u_m^{(-2)}=\frac{1}{8}u^-_{\alpha\mathring{A}}\tilde\sigma^{\alpha\beta}_m u^-_{\beta\mathring{A}},\quad u_m^{(+2)}=\frac{1}{8}u^+_{\alpha A}\tilde\sigma^{\alpha\beta}_m u^+_{\beta A}. \tag{2b}$$

On the equations of motion for the n-hedral coefficients of the world surface $\delta S_{D,N}/\delta\rho^\mu=0=\delta S_{D,N}/\delta\varphi^\mu$, the functional $S_{D,N}$ reproduces the action[8] $\tilde S_{D,N}=\frac{1}{2\alpha'}\int d\tau d\sigma\epsilon^{\mu\nu}\omega^m_\mu\omega^n_\nu u_m^{(-2)}u_n^{(+2)}$ with the auxiliary vectors $n^0_m=\frac{1}{2}(u_m^{(+2)}+u_m^{(-2)})\equiv u_m^{(0)}$, $n^1_m=\frac{1}{2}(u_m^{(+2)}-u_m^{(-2)})\equiv u_m^{(9)}$, constructed of Lorentz harmonics in accordance with (2). Since the formulation[8] with $N=0$ is equivalent to a Nambu–Goto string, this circumstance implies that the Lorentz-harmonic formulations in (1) are classically equivalent to corresponding versions of a Green–Schwarz superstring.

The Lorentz-harmonic variables[5] used in (2a) are bounded by the normalization (harmonicity) conditions $\Xi^{(4)}\equiv v^{\alpha-}v^+_\alpha-1=0$, $\overline{\Xi^{(4)}}\equiv\overline{v_\alpha}\bar v^{\dot\alpha+}-1=0$. When the invariance of (1) under gauge transformations from the group $[U(1)]^c=U_L(1)\times U_R(1)=SO(1,1)\times SO(2)$ is taken into account, these conditions make it possible to identify the space of harmonics $\{v_\alpha^\mp,v_\alpha^\pm\}$ with the $SL(2,C)/[U(1)]^c=SO(1,3)/SO(1,1)\times SO(2)$ factor space. Along with the components ρ^τ and φ^τ, four independent combinations of the harmonics $v_\alpha^\mp$ parameterize six independent components of the light-like vectors $\mathscr{P}_m\pm\frac{1}{\alpha}\omega_{\sigma m}+\cdots\equiv L_\pm$, which represent constraints in superstring theory.

Correspondingly, in (2b) $U^a_\alpha\equiv(u^+_{\alpha A},u^-_{\alpha\mathring{A}})$ [where $A=1,...,8$ and $\mathring{A}=1,...,8$ are (s)-and (c)-spinor indices of the $SO(8)$ group] are Lorentz-harmonic variables for $D=10$ (Ref. 6). They satisfy 1261 harmonicity conditions: $U^a_\alpha\tilde\sigma^{\alpha\beta}_{m_1\ldots m_5}U^b_\beta\sigma^{(n)}_{ab}=0$ and $\frac{1}{128}u^-_{\alpha\mathring{A}}\tilde\sigma^{\alpha\beta}_m u^-_{\beta\mathring{A}}u^+_{\gamma A}\tilde\sigma^{m\gamma\delta}u^+_{\delta A}=1$. These conditions impose only $210+1=211$ constraints on the Lorentz harmonics.[6] By virtue of the invariance of (1) under the $SO(1,1)\times SO(8)$ gauge group, only 16 of the 256 components U^a_α are independent ($16=256-211-28-1$). They may thus be regarded as the coordinates of a homogeneous $SO(1,9)/SO(1,1)\times SO(8)$ space. Along with ρ^τ and φ^τ, the 16 degrees of freedom of the harmonics U^a_α parameterize 18 independent components of two light-like vectors $L_\pm$. The quantity $S_{WZW,D,N}$ in (1) represents a Wess–Zumino–Witten term.[1] For a $D=4$, $N=1$ superstring, it is

$$S_{WZW,4,1}=\frac{1}{i\alpha'}\int d\tau d\sigma\epsilon^{\mu\nu}\partial_\mu x^m(\partial_\nu\theta^\alpha\sigma_{m\alpha\dot\alpha}\bar\theta^{\dot\alpha}-\theta^\alpha\sigma_{m\alpha\dot\alpha}\partial_\nu\bar\theta^{\dot\alpha}), \tag{3a}$$

while for a $D=10$, $N=2B$ superstring it is

$$S_{WZW,10,2B} = \frac{1}{i\alpha'} \int d\tau d\sigma \epsilon^{\mu\nu} [\omega_\mu^m \partial_\nu \Theta^{\alpha 1} \sigma_{m\alpha\beta} \Theta^{\beta 1}$$

$$+ \partial_\mu \Theta^{\alpha 1} \sigma_{m\alpha\beta} \Theta^{\beta 1} \partial_\nu \Theta^{\gamma 2} \sigma^m_{\gamma\delta} \Theta^{\delta 2}]. \tag{3b}$$

By virtue of the incorporation of this term, the action in (1) has κ symmetry. The generators of this symmetry are irreducible constraints of the first kind:

$$D=4: D^-(\sigma) \equiv v^{\alpha -} D_\alpha(\sigma) - \frac{4i}{\alpha' \rho^{(-1+)\tau}} \partial_\sigma \bar{\theta}^{\dot\alpha} \bar{v}^+_{\dot\alpha} \bar{\nabla}^{(-210)} + \frac{4i}{\alpha'} \partial_\sigma \bar{\theta}^{\dot\alpha} \bar{v}^-_{\dot\alpha} P^{(+1-)}_{(\rho)\tau},$$

$$D^+(\sigma) \equiv \overline{(D^-(\sigma))}, \quad (D_\alpha(\sigma) \equiv -\pi_\alpha + i\{P_m + \frac{1}{\alpha'} \omega_{\sigma m}\} \sigma^m_{\alpha\dot\alpha} \bar{\theta}^{\dot\alpha}), \tag{4a}$$

$$D=10: D^{1-}_A(\sigma) \equiv u^{\alpha -}_A D^1_\alpha - \frac{16i}{\alpha' \rho^{(+2)\tau}} \gamma^{(i)}_{A\mathring{A}} u^-_{\beta\mathring{A}} \partial_\sigma \Theta^{\beta 1} \nabla^{(+2)i} + \frac{4i}{\alpha'} u^+_{\beta A} \partial_\sigma \Theta^{\beta 1} P^{(-2)}_{(\rho)\tau},$$

$$D^{2+}_{\mathring{A}}(\sigma) \equiv u^{\alpha +}_{\mathring{A}} D^2_\alpha - \frac{16i}{\alpha' \varphi^{(-2)\tau}} \tilde{\gamma}^{(i)}_{\mathring{A}A} u^-_{\beta A} \partial_\sigma \Theta^{\beta 2} \nabla^{(-2)i} - \frac{4i}{\alpha'} u^-_{\beta\mathring{A}} \partial_\sigma \Theta^{\beta 2} P^{(+2)}_{(\varphi)\tau},$$

$$(D^1_\alpha \equiv -\pi^1_\alpha + i\{P_m - (-1)^1 \frac{1}{\alpha'} (\partial_\sigma X_m - i\partial_\mu \Theta^{\gamma 1} \sigma_{m\gamma\delta} \Theta^{\delta 1})\} \sigma^m_{\alpha\beta} \Theta^{\beta 1}). \tag{4b}$$

In (4b), $(u_A^{-\alpha}, u_{\mathring{A}}^{+\alpha}) \equiv U_a^{1\alpha}$ are components of the matrix which is the inverse of $U^a_\alpha \equiv (u^+_{\alpha A}, u^-_{\mathring{A}})$ (Ref. 6). Since the components $u_A^{-\alpha}$, $u_{\mathring{A}}^{+\alpha}$ cannot be expressed in a simple and covariant fashion in terms of $u^+_{\alpha A}$ and $u^-_{\alpha A}$, in contrast with the $D=4$ case, they should be regarded as independent harmonics. However, 256 invertibility conditions for the matrices u^a_α and $u_a^{-1\alpha}$: are also included in the number of harmonicity conditions: $(U_a^{-1\alpha} U^b_\alpha = \delta^b_a)$. The momentum densities $\mathscr{P}_{\mathscr{M}} \equiv -(-1)^{\mathscr{M}} \partial \mathscr{L} / \partial z^{\mathscr{M}}$, which are the canonical conjugates of the coordinates in target space[2], are denoted by $\mathscr{P}_{\mathscr{M}} \equiv (\mathscr{P}_m, \pi_\alpha, \bar{\pi}_{\dot\alpha}, P^{\pm}_\alpha, \bar{P}^{\mp}_{\dot\alpha}, P^{(+|-)}_{(\rho)\mu}, P^{(-|+)}_{(\varphi)\mu})$ and $\mathscr{P}_{\mathscr{M}} \equiv (\mathscr{P}_m, \pi^1_\alpha, P^{\alpha -}_A, P^{\alpha +}_{\mathring{A}}, P^+_{\alpha A}, P^-_{\alpha\mathring{A}}, P^{(-2)}_{(\rho)\mu}, P^{(+2)}_{(\varphi)\mu})$, respectively; $\bar{\nabla}^{(-210)} \equiv \bar{v}^{\dot\alpha -} \bar{P}^-_{\dot\alpha}$; $\nabla^{(-2)i} \equiv u^-_{\alpha\mathring{A}} \tilde{\gamma}^i_{\mathring{A}A} P^{\alpha -}_A - u_A^{-\alpha} \gamma^i_{A\mathring{A}} P^-_{\alpha\mathring{A}}$; $\nabla^{(+2)i} \equiv u^+_{\alpha A} \gamma^i_{A\mathring{A}} P^{\alpha +}_{\mathring{A}} - u_{\mathring{A}}^{+\alpha} \tilde{\gamma}^i_{\mathring{A}A} P^+_{\alpha A}$; and $\tilde{\gamma}^i_{\mathring{A}A} = (\gamma^i_{A\mathring{A}})$ and $\gamma^i_{A\mathring{A}}$ are $D=8$ σ matrices.[1]

We wish to stress that here, in contrast with superstrings, a massless superparticle[5] and null-super-p-branes[2] are described by only a single light-like momentum density vector $\mathscr{P}_m$, and their action is characterized by higher gauge symmetries $SO(1,1) \times SO(2) \times \mathscr{K}_2$ $(D=4)$ and $SO(1,1) \times SO(8) \times \mathscr{K}_8$ $(D=10)$, where $\mathscr{K}_2$ and $\mathscr{K}_8$ are Lorentz boosts generated by the operators[5,6] $\nabla^{(0|-2)} + ... \equiv v^{\alpha -} P^-_\alpha + ... (D=4)$ and $\nabla^{(-2)i} + ... \equiv u^-_{\alpha\mathring{A}} \tilde{\gamma}^i_{\mathring{A}A} P^{\hat\alpha}_A - u_A^{-\hat\alpha} \gamma^i_{A\mathring{A}} P^-_{\alpha\mathring{A}} + ... (D=10)$. Consequently, the harmonics $v^\pm_\alpha$ and $u^+_{\alpha A}, u^-_{\alpha\mathring{A}}$ can be thought of as the coordinates of homogeneous $SO(1,3)/SO(1,1) \times SO(2) \times \mathscr{K}_2 (D=4)$ and $SO(1,9)/SO(1,1) \times SO(8) \times K_8 (D=10)$ spaces. Along with ρ_τ, their number, $2(D=4)$ or $8(D=10)$, is exactly equal to the number of independent components of the light-like vector $\mathscr{P}_m$. Consequently, the requirement of a boost symmetry, which was imposed in a determination of Lorentz harmonics in Refs. 6 and 9, is permissible only for describing massless superparticles[5] and null-super-p-branes 2; it is not permissible for describing superstrings.

A procedure for a covariant BRST-BFV quantization of formulations (1) analogous to a quantization of null-super-p-branes[2,3] is presently being studied.

[1)] Here and below, a numeral inside parentheses as a superscript on a variable is the weight w with respect to transformations of the $SO(1,1)$ gauge group which (on the one hand) is a subgroup of the Lorentz group $SO(1, D-1)$ and (on the other) is identified with the structural group of the world sheet in this formulation of superstrings.

[2)] $[z^{\mathcal{M}}(\sigma), \mathcal{P}_{\mathcal{N}}(\sigma')]_p = -\delta^{\mathcal{M}}_{\mathcal{N}}\delta(\sigma-\sigma')$; $z^{\mathcal{M}}M \equiv (x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}, v_\alpha^{\mp}, \bar{v}_{\dot{\alpha}}^{\pm} \rho^{(-1+)\mu}, \varphi^{(+1-)\mu}$ for $D=4$, $N=1$; $z^{\mathcal{M}} \equiv (X^m, \theta^{\alpha 1}, u^{+}_{\alpha A}, u^{-}_{\alpha \dot{A}}, u_A^{-\alpha}, u_{\dot{A}}^{+\alpha}, \rho^{(+2)\mu}, \varphi^{(-2)\mu})$ for $D=10$, $N=2B$.

[1]M. B. Green, J. H. Schwarz, and E. Witten, *Superstring Theory, Vol. 1,* Cambridge Univ. Press, Oxford.
[2]I. A. Bandos and A. A. Zheltukhin, in *Proceedings of the Ninth International Conference on Problems of Quantum Field Theory,* Dubna, 1990; I. A. Bandos and A. A. Zheltukhin, Pis'ma Zh. Eksp. Teor. Fiz. **51**, 547 (1990) [JETP Lett. **51**, 618 (1990)].
[3]I. A. Bandos and A. A. Zheltukhin, Pis'ma Zh. Eksp. Teor. Fiz. **53**(1), 7 (1991) [JETP Lett. **53**, 5 (1991)].
[4]E. T. Newmann and R. Penrose, J. Math. Phys. **3**, 566 (1962).
[5]I. A. Bandos, Yad. Fiz. **51**, 1429 (1990) [Sov. J. Nucl. Phys. **51**, 906 (1990)]; Pis'ma Zh. Eksp. Teor. Fiz. **52**(4), 837 (1990) [JETP Lett. **52**, 205 (1990)].
[6]F. Deldus, A. Galperin, and E. Sokatchev, Preprint IMPERIAL/to/90–91/26, PARLPTHE/91–40, Imperial College, London–Paris, May, 1991.
[7]E. Nissimov, S. Pacheva, and S. Solomon, Nucl. Phys. B **296**, 469 (1988); **297**, 349 (1988); **299**, 183 (1988); **317**, 344 (1989).
[8]D. V. Volkov and A. A. Zheltukhin, Ukr. Fiz. Zh. **30**, 809 (1985).
[9]A. Galperin, P. Yjwe, and K. Stelle, Imperial College Preprint IMPERIAL, 1990–1991, London, 1991.

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