Remark on equivalence of topological and quantum 2D gravity

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We demonstrate the equivalence of Virasoro constraints imposed on the continuum limit of the partition function of the Hermitian 1-matrix model and the Ward identities of Kontsevich's model.

1. Since the early days of matrix models there has been the belief that partition functions of two *a priori* different models of 2D gravity should coincide. The first partition function is the (square root) of the partition function $\sqrt{Z_{qg}\{T_n\}}$ of ordinary Polyakov's quantum 2D gravity, which is described by the continuum limit of Hermitian 1-matrix model[1] (and is known to be a τ function of KdV hierarchy[2]). The second partition function Z_{tg} of Witten's topological gravity[3] is *a priori* defined as a generating functional of intersection indices of divisors on module spaces of Riemann surfaces with punctures.

Later, it was found[4,5] that $\sqrt{Z_{qg}\{T_n\}}$ satisfies a set of Virasoro constraints:

$$\mathscr{L}_n\sqrt{Z_{qg}\{T\}}=0, \quad n\geqslant -1 \tag{1}$$

$$\mathscr{L}_n=\sum_{k\geqslant\delta_{n+1,0}} T_k\frac{\partial}{\partial T_{k+n}}+\sum_{\substack{a+b=n1\\ a,b0}}\frac{\partial^2}{\partial T_a\partial T_b}+\delta_{n+1,0}\cdot\frac{T_0^2}{16}+\delta_{n,0}\cdot\frac{1}{16}.$$

Equation (1) can be deduced[6] as a continuum limit of Ward identities in a discrete matrix model, which is associated with the shift of the integration variables[7]

$$X\to X+\epsilon X^{p+1} \tag{2}$$

in the integral

$$Z_N^{(d)}\{t\}=\int DX\exp - trV\{X\} \tag{3}$$

$$V\{X\}=\sum_{k=0}^{\infty} t_k X^k.$$

However, it may be more reasonable to assume these Virasoro constraints to be a straightforward definition of $Z_{qg}\{T\}$, which does not refer to a sophisticated change of variables $\{t\}\to\{T\}^6$ and to a discrete model (3).

As for Z_{tg}, it was recently represented by Kontsevich[8] in terms of another matrix model:

$$Z_{tg} = \lim_{\text{size } X \to \infty} Z_{tg}^{(d)}, \tag{4}$$

where

$$Z_{tg}^{(d)} = \frac{1}{C[M]} \int DX \exp - tr\{MX^2 + X^3\}, \tag{5}$$

with

$$C[M] = \int DX \exp - tr\{MX^2\} = \det(M \otimes I + I \otimes M)^{-1/2}, \tag{6}$$

where X and M are (anti-)Hermitian matrices.

It is a simple combinatorial result that as soon as X goes to infinity, $Z_{tg}^{(d)}$ becomes dependent only on the variables

$$T_m = \frac{3^{2m+1}}{m+\frac{1}{2}} tr\, M^{-2m-1} + \frac{4}{3\sqrt{3}} \delta_{m,1}. \tag{7}$$

The explicit formulation of the original suggestion in these terms is

$$Z_{tg}\{T_m\} = \sqrt{Z_{qg}\{T_n\}}, \tag{8}$$

which, in particular, implies what is known as Witten's suggestion, i.e., that $Z_{tg}\{T\}$ is, like $\sqrt{Z_{qg}\{T\}}$, a τ function of KdV hierarchy. A necessary condition for (8) is that $Z_{tg}\{T\}$, which is defined by (4–6), satisfies the same Virasoro constraints (1).

This is the statement we prove in this letter. In addition, we shall prove a relation involving $Z_{tg}^{(d)}$ [see Eq. (16) below], i.e., valid for *finite* dimensional matrices X, which implies the entire set of Virasoro constraints only as the size of the matrix goes to infinity.

This statement is equivalent to (8) modulo:

(i) the assertions which have been made by Kontsevich in the derivation of (4)–(6) from Penner's formalism of fat graphs, and

(ii) the so-far subtle problem of the uniqueness of solutions of the Virasoro constraints (1).

2. Since the Ward identities are somewhat obscure in Kontsevich's presentation, (4)–(6), we begin with a slight reformulation of his model.

After a shift of the integration variable $X \to X - \frac{M}{3}$ and the redefinition $M = 3\Lambda^2$, we obtain

$$\mathscr{F}\{\Lambda\} \equiv \int DX \exp(-tr\, X^3 + tr\, \Lambda X)$$

$$= C[\sqrt{\Lambda}] \exp\left(-\frac{2}{3\sqrt{3}} tr\, \Lambda^{3/2}\right) Z_{tg}^{(d)} \tag{9}$$

with

$$C[\sqrt{\Lambda}] = \det(\sqrt{\Lambda} \otimes I + I \otimes \sqrt{\Lambda})^{-1/2}. \tag{10}$$

The functional $\mathcal{F}\{\Lambda\}$ can be considered as a kind of matrix-Fourier transform of the exponential cubic potential $tr\, X^3$. It satisfies obvious Ward identities which are associated with a shift of the integration variables:

$$X \to X + \epsilon_p. \tag{11}$$

Specifically,

$$tr\left(\epsilon_p \frac{\partial^2}{\partial \Lambda_{tr}^2} - \frac{1}{3}\epsilon_p \Lambda\right)\mathcal{F}\{\Lambda\} = 0. \tag{12}$$

Here ϵ_p represents any X-independent *matrix*, which may be diagonal simultaneously with Λ, e.g., $\epsilon_p = \Lambda^p$.

Let us substitute in (12) the functional $\mathcal{F}$ in the form prescribed by (9)

$$\mathcal{F}\{\Lambda\} = C[\sqrt{\Lambda}]\exp\left(-\frac{2}{3\sqrt{3}}\, tr\, \Lambda^{3/2}\right) Z\{T_m\}, \tag{13}$$

with

$$T_m = \frac{1}{m+\frac{1}{2}}\, tr\, \Lambda^{-m-1/2} + \frac{4}{3\sqrt{3}}\delta_{m,1}. \tag{14}$$

We assert that after the substitution of (13) into (12) it becomes

$$\sum_{n \geqslant -1} tr(\epsilon_p \Lambda^{-n-2}) \mathcal{L}_n Z = 0. \tag{15}$$

The identity

$$\frac{1}{\mathcal{F}}\, tr\left(\epsilon_p \frac{\partial^2}{\partial \Lambda_{tr}^2} - \frac{1}{3}\epsilon_p \Lambda\right)\mathcal{F} = \frac{1}{Z}\sum_{n \geqslant -1} tr(\epsilon_p \Lambda^{-n-2}) \mathcal{L}_n Z \tag{16}$$

is valid *for any size of the matrix* Λ, but only in the limit of infinitely large Λ:

(i) it is reasonable to substitute $Z_{tg}^{(d)}$ in (9) by Z_{tg}, which depends only on $tr\, \Lambda^{-q}$ with half-integer q's;

(ii) the T_m's in (14) are actually independent variables;

(iii) all the quantities

$$tr(\epsilon_p \Lambda^{-n-2}) \tag{17}$$

(e.g., $tr\, \Lambda^{p-n-2}$) become algebraically independent, so that Eq. (15) implies that

$$\mathcal{L}_n Z_{tg}\{T\} = 0, \quad n \geqslant -1. \tag{18}$$

This concludes the derivation of the Virasoro constraints for $Z_{tg}\{T\}$ defined as the continuum limit of Kontsevich's partition function.

The fact that the operators $\mathcal{L}_n$ in (15) contain only the second T derivatives is a direct result of the fact that only double Λ derivatives arise in the *l.h.s.* and thus the fact that the potential $tr X^3$ in Kontsevich's model has a cubic nature. It is very interesting (in particular, from the point of view of multimatrix and Potts models) to study the matrix-Fourier transform of the potentials like $tr X^{K+1}$ for any degree $K+1$. Interestingly, the corresponding Ward identities, which are transformed to the form (15), contain operators similar to the higher-spin operators of the W_K-algebra, instead of the Virasoro generators $\mathscr{L}_n$.

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