Trigonometric Lie subalgebras in $\bar{X}_\infty = \bar{A}_\infty$ (and, correspondingly, $\bar{B}_\infty$, $\bar{C}_\infty$, and $\bar{D}_\infty$) and their representations by vertex operators

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Infinite-dimensional trigonometric Lie subalgebras are introduced in $\bar{X}_\infty = \bar{A}_\infty$ (and, correspondingly, in $\bar{B}_\infty$, $\bar{C}_\infty$, and $\bar{D}_\infty$). These subalgebras are natural generalizations of the familiar trigonometric sine algebra. Explicit expressions for immersion in $\bar{X}_\infty$ are given. An irreducible representation of the $\bar{A}_\hbar$ and $\bar{B}_\hbar$ series is constructed in terms of vertex operators.

1. Kac-Moody algebras (the *A–D* series) are part of infinite-dimensional Lie matrix algebras $\bar{X}_\infty = \bar{A}_\infty$ ($\bar{B}_\infty$,$\bar{C}_\infty$,$\bar{D}_\infty$) and may be thought of as "periodic" reductions from $\bar{X}_\infty$ (Refs. 1 and 2). In terms of the theory of integrable equations, this immersion corresponds to the following situation: The Lie algebras $\bar{X}_\infty$ correspond to generalized two-dimensionalized Toda chains, and the *A–D* series of the two-dimensionalized chains are their periodic reductions.[3] The sine trigonometric Lie algebra provides an example of an "aperiodic" immersion in $\bar{A}_\infty$ (Ref. 4). The immersion was discovered in Ref. 5 (and rediscovered later[6] by a different approach; unfortunately, we were informed about Ref. 5, by A. N. Kirillov, only after our paper[6] was published). Curiously, the theory of integrable equations has an example of an "aperiodic" reduction from $\bar{A}_\infty$ of a generalized two-dimensionalized Toda chain to special integrable equations. This circumstance was the primary motivation for the determination and study of general classes of "trigonometric" Z-graduated subalgebras in $\overline{X}_\infty$ which generalize the Lie sine algebra.

Our algebras are examples of a broad class of continuum contragradient Lie algebras as introduced in Ref. 8. We would like to point out that algebras of this sort have arisen in gauge theories of higher spins,[9] and it would be useful to pursue this relationship explicitly.

2. We will apply the term "trigonometric algebras of the series $A_{\hbar}$, $\hbar = (\hbar_1,...,\hbar_k)$ to Lie algebras with generatrices $A_{\alpha,m}$ and c, where $\alpha = (\alpha_1,...,\alpha_k)$ is an integer vector of a k-dimensional plane $(\alpha,m)\in\mathbb{Z}^k\times\mathbb{Z}\setminus(0,...,0)$, and with the commutation relations

$$[A_{\alpha,m}, A_{\beta,l}] = 2i\sin[m(\hbar,\beta) - l(\hbar,\alpha)]A_{\alpha+\beta,m+l} + m\delta_{\alpha+\beta,\bar{0}}\delta_{m+l,0}c, \tag{1}$$

where c is the central element, and $(\hbar,\alpha) = \hbar_1\alpha_1 + ... + \hbar_k\alpha_k$. Lie algebra (1) can be realized as the cross product of an algebra of functions on a k-dimensional torus $T^k = \{\varphi = (\varphi_1,...,\varphi_k)|\varphi_k \bmod 2\pi\}$ by the shift operator $U = e^{2(\hbar_1\partial/\partial\varphi_1 + ... + \hbar_k\partial/\partial\varphi_k)}$. We assume $q = (q_1,...,q_k)$, where $q_l = e^{i\hbar_l}$. We also assume $q^\alpha = q_1^{\alpha_1}...q_k^{\alpha_k}$. The generatrices $A_{\alpha,m} = q^{m\alpha}e^{i(\alpha,\varphi)}U^m$ then satisfy (1) with $c = 0$. We denote Lie algebra (1) with $c = 0$ by $A_{\hbar}$. Then $\overline{A}_{\hbar}$ is a one-dimensional central expansion of $A_{\hbar}$.

3. Following Ref. 10, we choose in $\overline{A}_{\hbar}$ a Heisenberg subalgebra $\vec{\alpha}S = \{A_{\bar{0},m}|m\in\mathbb{Z}\setminus 0\}$ and a maximal commutative subalgebra (a Cartan subalgebra) $H = \{A_{c,0}|\alpha\in\mathbb{Z}^k\setminus\bar{0}\}$. We define the fields $X_\alpha(z) = \Sigma_{l\in\mathbb{Z}} A_{\alpha,l}z^{-l}$, where $\alpha\in\mathbb{Z}^k\setminus\bar{0}$, and z is a complex variable. All the generatrices of the algebra $A_{\hbar}$ are contained in $\vec{S}$ and the fields $X_\alpha(z)$. It is easy to verify that the following relations holds:

$$[A_{\bar{0},m}, X_\alpha(z)] = 2i\sin(m(\hbar,\alpha))z^m X_\alpha(z), \quad m\in\mathbb{Z}\setminus 0. \tag{2}$$

The Heisenberg algebra $\vec{S}$ has a standard irreducible representation in the space $V = C[x_1,x_2,...]$:

$$\pi_0(A_{\bar{0},m}) = \partial/\partial x_m; \quad \pi_0(A_{0,-m}) = mx_m; \quad \pi_0(c) = 1, \quad m > 0.$$

Equations (2), in which $A_{\bar{0},m}$ are represented by the operators $\partial/\partial x_m, mx_m$, have a unique solution (within multiplication by a constant a_α) in the class of differential operators which act on the space V (Ref. 10):

$$\hat{X}_\alpha(z) = a_\alpha \exp(2i\sum_{m\geq 1} z^m\sin(m(\hbar,\alpha))x_m)\exp(2i\sum_{m\geq 1}\frac{z^{-m}}{m}\sin(m(\hbar,\alpha))\partial/\partial x_m. \tag{3}$$

We introduce the generatrices $\hat{X}_{\alpha,l} = 1/2\pi i \oint_\Gamma dz z^{l-1}\hat{X}_\alpha(z)$, where the integration is over a contour Γ which loops the point 0. The correspondence

$$\pi(A_{\alpha,l}) = \hat{X}_{\alpha,l}, \quad \alpha\neq\bar{0};$$

$$\pi(A_{0,l}) = \partial/\partial x_l; \quad \pi(A_{0,-l}) = lx_l, \quad \pi(c) = 1, \quad l > 0$$

then specifies an irreducible representation of the senior weight $\Lambda\in H^*$, where $\Lambda(A_{\alpha,0})$

$= a_\alpha$, in space V with a vacuum vector $|0\rangle = 1$. The constants a_α are determined unambiguously within a phase factor $\exp(\lambda,\alpha)$, $\lambda = (\lambda_1,...,\lambda_k)$. They are given by $a_\alpha = q^\alpha/(q^\alpha - q^{-\alpha})$. This fact can be proved by a method like that of Ref. 6.

4. It is not difficult to see that the vertex operators in (3) can be found from the vertex operator[1] $\mathbb{Z}(u,v)$, which realizes a basis representation of the Lie algebra $\overline{A}_\infty = \widehat{Gl}(\infty)$, by means of the reduction $u = zq^\alpha$, $v = zq^{-\alpha}$. This circumstance motivates the following assertion, which can be verified directly. We assume that $E_{i,j}, i,j \in Z$ satisfy the commutation relations $\widehat{Gl}(\infty): [E_{i,j}, E_{k,l}] = E_{i,l}\delta_{j,k} - E_{k,j}\delta_{k,l} + \psi(E_{i,j}, E_{k,l})$, where the 2-cocycle ψ on the Lie algebra $Gl(\infty)$ is defined by the conditions

$$\psi(E_{i,j}, E_{j,i}) = 1 = -\psi(E_{ji}, E_{ij}), \quad \text{if} \quad i \le 0, \; j \ge 1$$

$$\psi(E_{i,j}, E_{k,l}) = 0 \; \text{ otherwise.} \tag{4}$$

The explicit expressions for the immersion of $\overline{A}_\hbar$ in $\overline{A}_\infty$ are then

$$A_{\alpha,m} = q^{m\alpha} \sum_{n\in Z} q^{2n\alpha} E_{n,n+m} + \delta_{m,0} a_\alpha c, \tag{5}$$

where $a_\alpha = q^\alpha/(q^\alpha - q^{-\alpha})$. We note that (5) is a "maximal" generalization of the corresponding immersion formula from Refs. 5 and 6.

5. We recall[2] that the subalgebras B_∞, C_∞, and D_∞ are defined as subalgebras in A_∞ which conserve the following bilinear forms, respectively:

$$< e_i, \; e_j >= (-1)^i \delta_{i,-j} \quad \text{in the case of} \quad B_\infty,$$
$$< e_i, \; e_j >= (-1)^i \delta_{i,1-j} \quad \text{in the case of} \quad C_\infty,$$
$$< e_i, \; e_j >= \delta_{i,1-j} \quad \text{in the case of} \quad D_\infty.$$

A one-dimensional central expansion of these algebras is specified by the 2-cocycle $r\psi$, where ψ is found from (4), and we have $r = 1/2$ for $\overline{B}_\infty$ and $\overline{D}_\infty$ and $r = 1$ for $\overline{C}_\infty$ (Ref. 2). It is then natural to determine the trigonometric algebras of the series $\overline{B}_\hbar$, $\overline{C}_\hbar$, and $\overline{D}_\hbar$; as one-dimensional central expansions by means of the cocyle $r\psi$ of the intersections of $A_\hbar$ with B_∞, C_∞, and D_∞. Direct calculations lead to the following list of results.

A trigonometric basis can be selected in the Lie algebras $B_\hbar$, $C_\hbar$, and $D_\hbar$:

$$B_{\alpha,m} = A_{\alpha,m} - (-1)^m A_{-\alpha,m} \quad \text{in } B_\hbar ,$$

$$C_{\alpha,m} = A_{\alpha,m} - (-1)^m q^{2\alpha} A_{-\alpha,m} \text{ in } C_\hbar ,$$

$$D_{\alpha,m} = A_{\alpha,m} - q^{2\alpha} A_{-\alpha,m} \quad \text{in } D_\hbar .$$

These subalgebras are described as fixed points of second-order automorphisms:

$$\text{for } B_\hbar \quad \tau_1(A_{\alpha,m}) = -(-1)^m A_{-\alpha,m} \quad ,$$
$$\text{for } C_\hbar \quad \tau_2(A_{\alpha,m}) = -(-1)^m q^{2\alpha} A_{-\alpha,m} \, ,$$
$$\text{for } D_\hbar \quad \tau_3(A_{\alpha,m}) = -q^{2\alpha} A_{-\alpha,m}.$$

The commutation relations which determine $\overline{B}_\hbar$, $\overline{C}_\hbar$, and $\overline{D}_\hbar$ can be calculated directly; we will not reproduce them here.

The formulas for immersion of $\overline{B}_\hbar$ in B_∞ are

$$B_{\alpha,m} = q^{m\alpha} \sum_{n\in Z}{}' q^{2n\alpha}(E_{n,n+m} - (-1)^m E_{-n-m,-n}) + \delta_{m,0} b_\alpha c, \tag{6}$$

where $b_\alpha = 1/2(q^\alpha + q^{-\alpha})/(q^\alpha - q^{-\alpha})$. The formula for immersion of $\overline{C}_\hbar$ in $\overline{C}_\infty$ is

$$C_{\alpha,m} = q^{m\alpha} \sum_{n\in Z}{}' q^{2n\alpha}(E_{n,n+m} - (-1)^m E_{1-n-m,1-n}) + 2\delta_{m,0} a_\alpha c. \tag{7}$$

The formula for the immersion of $\overline{D}_\hbar$ in $\overline{D}_\infty$ is

$$D_{\alpha,m} = q^{m\alpha} \sum_{n\in Z}{}' q^{2n\alpha}(E_{n,n+m} - E_{1-n-m,1-n}) + \delta_{m,0} a_\alpha c. \tag{8}$$

Formulas (5)–(8) can be used to construct a representation of the Lie algebras $A_\hbar - D_\hbar$ in the space $C^\infty(S^1)$, of complex-valued functions on the circle $S^1 = \{\varphi_1 \bmod 2\pi\}$:

$$A_{\alpha,m} \to q^{-m\alpha} e^{-im\varphi_1} e^{-2(\hbar,\alpha)\partial/\partial\varphi_1}.$$

In the case of the series $\overline{B}_n$, it is also possible to construct a basis representation in terms of vertex operators. The corresponding operators are found from $(1/2)(u-v)/(u+v)(\Gamma_B(u,v)-1)$ (p. 228 in Ref. 2) by means of the ansatz $u = zq^\alpha$, $v = -zq^{-\alpha}$.

6. Let us look at some particular cases.

a) We assume $\hbar = \hbar_1$ and $\hbar_1 \notin \pi\mathbb{Q}$. Then $\overline{A}_{\hbar_1}$ is isomorphic to the sign algebra,[1] which is a Weyl quantization of the algebra $C^\infty(T^2)$, of functions on the two-dimensional torus $T^2 = \{(\varphi_1,\varphi_2) \bmod 2\pi\}$ with Poisson brackets. The subalgebras $B_{\hbar_1}$ and $C_{\hbar_1}$ in the classical limit then give us a subalgebra in $C^\infty(T^2)$ which consists of functions that have the additional symmetry $f(\varphi_1+\pi, -\varphi_2) = -f(\varphi_1,\varphi_2)$. The algebra $D_{\hbar 1}$ is a quantization of a Poisson subalgebra in $C^\infty(T^2)$ with the additional condition $f(\varphi_1, -\varphi_2) = -f(\varphi_1,\varphi_2)$.

b) We assume $\hbar = (\pi/N, \hbar_1)$, $\hbar_1 \notin \pi\mathbb{Q}$. We first consider the series $\overline{A}_\hbar$. By virtue of (5), the generatrices $A_{\alpha,m}$ satisfy the auxiliary relations $A_{n_1+rN,n_2,m} = (-1)^{mr} A_{n_1,n_2,m}$, $r\in\mathbb{Z}$. We denote by $\overline{A}_{N,\hbar_1}$ a factor algebra in terms of these relations. In the limit $\hbar_1 \to 0$ this algebra is the same as the Kac-Moody Lie algebra of the $\overline{A}^{(1)}_{N-1}$ series. Consequently, $\overline{A}_{N,\hbar_1}$ may be thought of as a quantum deformation of $\overline{A}^{(1)}_{N-1}$.

Analogously, the Lie algebras $\bar{B}_{N,\hbar_1}$, $\bar{C}_{N,\hbar_1}$, and $\bar{D}_{N,\hbar_1}$ which we have determined have the following limits as $\hbar_1 \to 0$: $\bar{B}_{N,\hbar_1}$ with $N = 4l$ becomes $\bar{D}_{2l}^{(1)}$; $\bar{C}_{N,\hbar_1}$ with $N = 2l$ becomes $\bar{C}_l^{(1)}$; $\bar{D}_{N,\hbar_1}$ with $N = 2l$ becomes $\bar{D}_l^{(1)}$; and with $N = 2l$ it instead becomes $\bar{B}_l^{(1)}$.

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