Minimum metallic conductivity in the theory of localization

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An Anderson metal-insulator transition is examined in the Cayley tree model. It is shown that the conducting region has a minimum metallic conductivity, while the localized region has a maximum dielectric constant.

No one now doubts the existence of an Anderson transition[1] in the insulator state with an increase of disorder in the metal. However, the question of the nature of the transition has given rise to much discussion. According to Mott,[2] the conductivity of the metal decreases with increasing disorder and reaches the minimum value $\sigma_{\min} \sim e^2/\hbar a$, where a is the interatomic distance. As the disorder increases further, the

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conductivity vanishes in a jump-like manner, and the system becomes an insulator. According to the opposite point of view, the conductivity cannot have jumps, but vanishes smoothly at the point of the transition. After the appearance of the renormalization-group technique,[3] the second point of view was generally accepted. Only the question of the magnitude of the indices in the power-law diminution of the conductivity remained. There were no special disagreements concerning the behavior of the dielectric constant as the point of the transition was approached from the insulator side. It was assumed that it becomes infinite.

We will examine below a model of a disordered metal which allows an exact solution and in which a metal-insulator transition occurs. An unexpected result of the model is the existence of a minimum metallic conductivity in the metallic region and a maximum dielectric constant in the localized region.

We will examine a system comprised of separate metallic granules in contact with each other. Electrons can tunnel from one group to another. Each granule contains randomly positioned impurities. It is assumed that the mean free path in the granules is much greater than the interatomic distance. The kinetics of the electrons in the random potential is entirely described by the density-density correlation function, which can be calculated by the method of supersymmetry.[4] After some transformations typical for this method, we can write the density-density correlation function $K(r,r')$ in the form

$$K(r, r') = -2\pi^2\nu^2 \int Q_{13}^{12}(r) Q_{31}^{21}(r') \exp(-F[Q]) \prod_i dQ_i , \tag{1}$$

where

$$F[Q] = \sum_{i,j} J_{ij} \,\mathrm{STr}\,(Q_i - Q_j)^2 + \frac{i(\omega + i\delta)\pi\nu}{4} \sum_j \mathrm{STr}\,(\Lambda Q_j)\, V_j .$$

Equation (1) is an integration over the supermatrix Q. The volume of the j-th granule is V_j, and the state density is ν. The quantities J_{ij} are proportional to the square of the ratio of the amplitude of tunneling from one granule into another to the average distance between the levels in the granules. The 8×8 Q and Λ matrices are

$$Q = U \times \begin{pmatrix} \cos\hat\theta & i\sin\hat\theta \\ -i\sin\hat\theta & -\cos\hat\theta \end{pmatrix} \bar U, \quad U = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

The U and $\hat\theta$ matrices are described in greater detail in Ref. 4. We note only that $\bar U U = 1$. The indices in the matrix U in (1) determine certain elements of these matrices. the symbol STr denotes a supertrace. The model defined by Eqs. (1) and (2) cannot, in general, be solved for an arbitrary lattice of granules in a space of arbitrary dimension. In a space of low dimensionality, only the slow spatial changes in Q are important. In this limit the finite differences are replaced by spatial derivatives, and we obtain the nonlinear σ model,[4] which is studied in connection with the study of localization in a space of low dimensionality. In the case of the Anderson transition we will examine the simplest case of identical bonds and identical granules that form a Bethe lattice (Cayley tree) with m branches. In the theory of phase transitions, Cayley tree models essentially coincide with the self-consistent-field approximation. In the theory of localiztion, a mean-free theory has not yet been developed. One can hope that, as in the theory of phase transitions, the Cayley tree model will give a good qualitative description of the transition. We note that the model under examination differs from the Anderson Cayley tree model, studied in Ref. 5. In Ref. 5, the existence of a transition was proved for specific distributions of the potential, but the nature of the critical behavior was not explained. In Ref. 6 it was assumed that the conductivity in the Cayley tree decreases in a power-law fashion as the transition point is approached. In deriving this result, rather strong assumptions were made in Ref. 6. In particular, the resistance along the Cayley tree was calculated and it was assumed that the resistivity behaves just like the total resistivity. The results obtained below disagree with the assertion of a power-law decrease. The question as to how the resistance along the Cayley tree should be understood does not arise at all, since the density-density correlation function, whose form determines directly the diffusion coefficient, is calculated.

The structure of the Cayley tree permits using the transition-matrix method, which is analogous to the method used for one-dimensional systems. The correlation function can easily be calculated at the coinciding points. This correlation function makes it possible to distinguish the conducting region from the localized region and to calculate the diffusion coefficient and dielectric constant. Assuming that only the nearest neighbors interact, we can reduce the calculation of the correlation function $K(r,r)$ to the calculation of an integral from the solution of the integral equation

$$K(r,r) = -2\pi^2\nu^2 \int Q_{13}^{12} Q_{31}^{21} \psi^{m+1}(Q) \exp\left(\frac{\beta}{4} \operatorname{STr}(\Lambda Q)\right), \tag{3}$$

where $(\beta = -i(\omega + i\delta)\pi\nu V)$.

The function ψ satisfies the integral equation

$$\psi(Q) = \int \exp\left(-\operatorname{STr}\left(\frac{\alpha}{8}(Q-Q')^2 - \frac{\beta}{4}\Lambda Q'\right)\right)\psi^m(Q')dQ'. \tag{4}$$

In Eq. (4) the quantity α is related to the interaction of nearest neighbors J by the relation $\alpha = 8J$.

The integration over all supermatrices Q of the form (2) is performed in expressions (3) and (4). The case $m = 1$ corresponds to the one-dimensional chain of granules. At $\alpha \gg 1$ it is the model of a disordered wire. In this limit the integral equation becomes a differential equation. The corresponding calculations were performed in Ref. 7, where localization is proved and the dielectric constant is calculated.

The symmetry of the supermatrix Q depends on the magnetic and spin-orbit interactions. The simplest calculations are those when there are magnetic interactions and there are no spin-orbit interactions. The matrix θ in (2) in this case is

$$\hat{\theta} = \begin{pmatrix} \theta & 0 \\ 0 & i\theta_1 \end{pmatrix}. \tag{5}$$

As in Ref. 7, we seek a solution that depends only on the variables θ_1 and θ.

Integrating over all remaining variables in (3) and (4), we find

$$K(r,r) = \pi^2 \nu^2 \int_{-1}^{1} \int_{1}^{\infty} \frac{\lambda_1 + \lambda}{\lambda_1 - \lambda} \psi^{m+1}(\lambda_1, \lambda) e^{\beta(\lambda - \lambda_1)} d\lambda d\lambda_1 \qquad (6)$$

$$\frac{\psi(\lambda, \lambda_1) - 1}{\lambda_1 - \lambda} = \int_{-1}^{1} \int_{1}^{\infty} L(\lambda, \lambda_1; \lambda', \lambda'_1) \frac{(e^{\beta(\lambda' - \lambda'_1)} \psi^m(\lambda', \lambda'_1) - 1)}{\lambda'_1 - \lambda'} d\lambda' d\lambda'_1, \qquad (7)$$

where

$$L(\lambda, \lambda_1; \lambda', \lambda'_1) = \frac{\alpha^2}{2} \int_0^{2\pi} \int_0^{2\pi} \left[\left(\frac{d}{d\alpha} e^{\alpha \mathbf{n}\mathbf{n}'} \right) e^{-\alpha \mathbf{n}_1 \mathbf{n}'_1} - e^{+\alpha \mathbf{n}\mathbf{n}'} \frac{d}{d\alpha} (e^{-\alpha \mathbf{n}_1 \mathbf{n}'_1}) \right]$$

$$\times \frac{d\varphi' d\varphi'_1}{(2\pi)^2},$$

$\mathbf{n} = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta)$, $\mathbf{n}' = (\sin\theta' \cos\varphi', \sin\theta' \sin\varphi', \cos\theta')$,

$\mathbf{n}_1 = (i \sinh\theta_1 \cos\varphi_1, i \sinh\theta_1 \sin\varphi_1, \cosh\theta_1)$,

$\mathbf{n}'_1 = (i \sinh\theta'_1 \cos\varphi'_1, i \sinh\theta'_1 \sin\varphi'_1, \cosh\theta'_1)$,

$\lambda = \cos\theta$, $\lambda_1 = \cosh\theta_1$, $\lambda' = \cos\theta'$, $\lambda'_1 = \cosh\theta'_1$.

Expressions (6) and (7) completely solve the problem of finding the density-density correlation function. The integration over φ' and φ_1 in the expression for L can be performed directly. However, this leads to more cumbersome expressions that contain Bessel functions. An analysis of Eq. (7) and calculation of integral (6) leads to the conclusion that at $\alpha > \alpha_c$, where α_c is the critical value, the system is conducting. At the critical point α_c, the conductivity vanishes in a jump-like manner. The magnitude of the jump can be determined only numerically. At $\alpha < \alpha_c$, the conductivity is zero. In this region, the dielectric constant increases with increasing α and attains a maximum value at α_c. For arbitrary values of m the magnitude of the maximum dielectric constant can also be determined only numerically. The quantity α_c is the solution of the equation

$$m \sqrt{\frac{2\alpha_c}{\pi}} \left[\left(\cosh\alpha_c - \frac{1}{2} \frac{\sinh\alpha_c}{\alpha_c} \right) K_0(\alpha_c) + \sinh\alpha_c K_1(\alpha_c) \right] = 1, \qquad (8)$$

where K_0 and K_1 are modified Bessel functions.

The analytic expressions for the density-density correlation function for arbitrary m can be found only in the limits $\alpha \ll \alpha_c$ and $\alpha \gg \alpha_c$. At $\alpha \gg \alpha_c$ the correlation function $K(r,r)$ is

$$K(r,r) = \frac{\pi^2 \nu^2}{D}, \qquad D = \frac{m^2 - 1}{m} \alpha. \qquad (9)$$

In the opposite limit, the case $\alpha \ll \alpha_c$, the corresponding expression is

$$K(r,r) = \frac{\pi\nu}{-i(\omega+i\delta)}, \quad \frac{1}{\kappa V}, \quad \kappa = 1 + \frac{1}{2}(m+1)\sqrt{\frac{\alpha\pi}{2}} + \frac{\alpha(m+1)m}{8}\ln(1/\alpha). \tag{10}$$

The quantities D and κ in (9) and (10) are proportional to the diffusion coefficient and the dielectric constant, respectively. We note that at $\alpha \ll \alpha_c$ the α dependence of κ is nonanalytic.

The conclusion that a minimum metallic conductvity exists disagrees sharply with the renormalization-group predictions. In the opinion of this author, this disagreement is not a result of some particular features of the Cayley tree model. If this assertion is correct, then the basic assumptions of the renormalization-group method must be reexamined. A more detailed paper on this subject will be published elsewhere.

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