Finiteness of $N = 4$ supersymmetry sigma models

Ya. I. Kogan, A. Yu. Morozov, and A. M. Perelomov
Institute of Theoretical and Experimental Physics

(Submitted 23 May 1984)
Pis'ma Zh. Eksp. Teor. Fiz. **40**, No. 1, 38–41 (10 July 1984)

A 2-D supersymmetry K3 sigma model cannot be renormalized in all orders of perturbation theory. A case is made for the vanishing of the β function for all $N = 4$ supersymmetric 2-D sigma models.

1. The four-dimensional ($N = 4$) supersymmetric Yang-Mills theory is finite.[1] A natural generalization of this assertion is the hypothesis that all theories with a maximal supersymmetry, in particular, the $N = 8$ supergravity, are finite. This hypothesis can be tested in some simpler cases, primarily the 2-D sigma models. The maximal allowed supersymmetry for σ models is $N = 4$, which is realized on hyper-Kähler manifolds M, which are necessarily Ricci-planar.[2] The Lagrangian of a Kähler σ model ($N = 2$ supersymmetry) is

$$\frac{1}{\alpha}\int g_{\alpha\bar{\beta}}(\Phi, \bar{\Phi}) D\,\Phi^{\alpha}\overline{D\,\Phi^{\beta}}\, d^2z\,d^2\theta\ ;$$

$$D = \frac{\partial}{\partial\bar{\theta}} - i(\sigma\theta)\frac{\partial}{\partial z}\ ;\quad \Phi^{\alpha}(x, \theta) = \phi^{\alpha}(x) + \bar{\theta}\psi^{\alpha}(x) + \frac{1}{2}\bar{\theta}\theta F^{\alpha}(x). \qquad (1)$$

The boson fields $\phi^{\alpha}(z, \bar{z})$ are coordinates on the manifold M with the Kähler metric $g_{\alpha\bar{\beta}}(\phi, \bar{\phi})$, while their superpartners $\psi^{\alpha}(z, \bar{z})$ lie in a space tangent to M[3]; here α takes on values from 1 to $\dim_C M$. Manifolds with a hyper-Kähler metric[2] exist only for even complex dimensionalities of M; the simplest examples are found with $\dim_K M = 4$. These are asymptotically locally Euclidean gravitational instantons and manifolds of the K3 type.[4] The former determine noncompact, and the latter compact, σ models. Alvarez-Gaumé and Freedman[2] have asserted that the β function vanishes for the $N = 4$ supersymmetry σ models, and they attempted to prove this assertion for gravitational-instanton models. As we understand it, their proof is based on the concept of a conformal weight and thus uses the nonobvious assumption that there are no tensors with a nonvanishing conformal weight which are constant on M. In the present letter we offer some slightly different arguments, which require more-detailed knowledge of Ricci-planar manifolds. We will pursue the arguments completely for the K3 models, but the arguments can be generalized to other cases.

2. The quantum corrections to the action (1) which conserve the $N = 2$ supersymmetries are $\int T_{\alpha\bar{\beta}}(\Phi, \bar{\Phi})D\Phi^{\alpha}\,\overline{D\Phi^{\beta}}\,d^2z\,d^2\theta$. Here $T_{\alpha\bar{\beta}}$ is a Kähler tensor, which is expressed in a single loop in terms of the metric:

$$T^{(1)}_{\alpha\bar{\beta}} \sim \partial_{\alpha}\partial_{\bar{\beta}}\,|\ln\det g_{\mu\bar{\nu}}| \sim R_{\alpha\bar{\beta}}\ . \qquad (2)$$

In higher orders, it is expressed in terms of the scalar S, which depends on the curvature tensor and its covariant derivatives:

$$T^{(2,\ldots)}_{\alpha\bar\beta} = \partial_\alpha \partial_{\bar\beta} S^{(2,\ldots)} \qquad \left(\partial_\alpha = \frac{\partial}{\partial\phi^\alpha}\,;\quad \partial_{\bar\beta} = \frac{\partial}{\partial\bar\phi^\beta}\right). \tag{3}$$

For example,[5] $T^{(2)}_{\alpha\bar\beta} \sim \partial_\alpha \partial_{\bar\beta} R$. It is important to note that these expressions do not depend on the particular choice of the manifold M, and, beginning at two loops, the quantum corrections are expressed in terms of second derivatives of nonsingular scalars S, which are invariants of coordinate-independent transformations on M. The auxiliary conservation condition of the $N=4$ supersymmetry incorporates the Ricci planarity of the metric $g_{\alpha\bar\beta} + T_{\alpha\bar\beta}$; i.e., $\partial_\alpha \partial_{\bar\beta} \ln \det(g+T) = 0$. In Section 3 we show which conditions arise on $T_{\alpha\bar\beta}$ in this case, and in Section 4 we prove that in the K3 case it follows from these conditions that there are no corrections of the type in (3). The single-loop correction (2) is proportional to the Ricci tensor and itself vanishes.

3. The condition for Ricci planarity of the metric $g+T$ (in the real representation) is $R_{ij}(g_{kl}+T_{kl})=0$. We then immediately find an equation for T_{ij}: $\Delta_L T_{ij} \equiv \nabla^k \nabla_k T_{ij} + [\nabla_i, \nabla_k] T^k_j + [\nabla_j, \nabla_k] T_i^{\cdot k} = 0$. Working from the equation for the commutator of the covariant derivatives and from the condition $R_{ij}=0$, we find the expression $\Delta_L T_{ij} = \nabla^k \nabla_k T_{ij} - 2R_{ikjl} T^{kl} = 0$. In the Kähler case we have $(\nabla^\alpha \nabla_\alpha) T_{\beta\bar\gamma} + [\nabla_\gamma, \overline{\nabla^\delta}] T_{\beta\bar\delta} + [\nabla_\beta, \nabla^\delta] T_{\delta\bar\gamma} = 0$. We have the following useful identity:

$$\frac{1}{2} \Delta_L T_{\beta\bar\gamma} = \nabla^\alpha (\nabla_\alpha T_{\beta\bar\gamma} - \nabla_\beta T_{\alpha\bar\gamma}) + \nabla_\beta \nabla^\alpha T_{\alpha\bar\gamma}. \tag{4}$$

We introduce in the space of tensors a positive-definite scalar product in the standard fashion: $\langle T_{\alpha\bar\beta}, T_{\alpha\bar\beta}\rangle \equiv \int T^*_{\alpha\bar\beta} T_{\alpha\bar\beta} d\mu(\phi)$. From (4) we then find

$$-\langle T_{\alpha\bar\beta}, \Delta_L T_{\alpha\bar\beta}\rangle = \langle \nabla_\alpha T_{\beta\bar\gamma} - \nabla_\beta T_{\alpha\bar\gamma}, \times\, \nabla_\alpha T_{\beta\bar\gamma} - \nabla_\beta T_{\alpha\bar\gamma}\rangle + 2\langle \nabla^\alpha T_{\alpha\bar\gamma}, \nabla^\beta T_{\beta\bar\gamma}\rangle. \tag{5}$$

The condition $\Delta_L T_{\alpha\bar\beta} = 0$ is therefore equivalent to the two following conditions:

$$\nabla_\alpha T_{\beta\bar\gamma} - \nabla_\beta T_{\alpha\bar\gamma} = \partial_\alpha T_{\beta\bar\gamma} - \partial_\beta T_{\alpha\bar\gamma} = 0 + \text{the complex-conjugate condition}, \tag{6}$$

$$\nabla^\alpha T_{\alpha\bar\gamma} = 0 + \text{the complex-conjugate condition}. \tag{7}$$

Condition (6) is simply the condition that the tensor $T_{\alpha\bar\beta}$ is a Kähler tensor. From (6) we then find $\nabla^{\bar\gamma} T_{\beta\bar\gamma} = \nabla_\beta (T^{\bar\gamma}_{\bar\gamma}) = \partial_\beta (T^\gamma_\gamma)$. By virtue of (7), however, we have $\nabla^{\bar\gamma} T_{\beta\bar\gamma} = 0$ and $\partial_\beta T^\gamma_\gamma = 0$. It follows that $T^\gamma_\gamma = \text{const}$. We will make use of this condition below in our study of the K3 case.

4. The K3 manifold is a singly connected complex manifold with the first Chen class, $C_1 = 0$. According to the Yau theorem,[6] all the K3 manifolds therefore allow a Ricci-planar Kähler metric $g_{\alpha\bar\beta}$. It has recently been shown that such a metric always exists on a K3 manifold. The holonomic group of such a metric reduces from $SO(4) = SO(3) \times SO(3)$ to $SO(3) \simeq Sp(1)$; i.e., the metric allows a hyper-Kähler structure. We know that on these manifolds there exists a finite number of algebraic 2-cycles $\sim CP_1$ (F. Bogomolov, private communication), so that instanton solutions exist in 2-D

K3 σ models (these solutions may prove useful; see Section 5). We know that the Euler characteristic of the K3 manifold is $\chi = 24$; i.e., the number of harmonic 2-forms is $b_2 = \chi - 2 = 22$. The Hirzebruch signature is $\tau = b_2^+ - b_2^- = 16$. We thus have $b_2^+ = 19$ self-dual harmonic 2-forms and $b_2^- = 3$ anti-self-dual harmonic 2-forms. Of these, 20 forms are of the (1,1) type, one is of the (2,0) type, and one is of the (0,2) type; the last two are anti-self-dual forms (the type of form is determined in the spinor representation). All this information has been taken from the paper by Page.[7] Second-rank Kähler tensors satisfying the equation $T^\gamma_\gamma = 0$ are constructed by multiplying self- and anti-self-dual forms of the (1,1) type. In all, there are 19 such products, and all the tensors $T_{\alpha\bar\beta}$ determined in this manner are traceless. From definition (3) of the tensor $T_{\alpha\bar\beta}$ we find the equation $T^\gamma_\gamma = \Delta S = 0$. This equation does not have solutions which are different from a constant in the class of *nonsingular* scalars. With $S =$ const we find a vanishing tensor $T_{\alpha\bar\beta}$. This point is clear from the outset. All the $T_{\alpha\bar\beta}$ constructed in this manner are related to the homotopic nontriviality of the K3 manifold and correspond to singular scalars. It is important to note that the perturbative corrections to the sigma models are universal and "are unaware of" the nontrivial structure of the manifold.

In general, solutions with $T^\gamma_\gamma = \text{const} \neq 0$ are possible, but, as Hitchin has shown,[7] there is only a single solution of the equation $\Delta_L T_{\alpha\bar\beta} = 0$ in the case $T^\gamma_\gamma \neq 0$. This solution is of course $cg_{\alpha\bar\beta}$. However, $g_{\alpha\bar\beta}$ is a second derivative of a Kähler potential, which we know cannot be a function which is globally defined on K3 manifold. This completes the proof of the finiteness of the K3 sigma model.

In the following section we will assume that the only possible renormalization in the effective Lagrangian reduces to a change in the coefficient of the seed action (1).

5. What happens to the instanton calculation of the β function[8,9] in the case of Ricci-planar manifolds M? In this case the β function is expressed in terms of the number of zero boson modes, n_B, and fermion modes, n_F, in the instanton field. Under the condition $n_F = n_B$, which holds for homogeneous Kähler manifolds,[9] the β function has only a single-loop component; the zero β function corresponds to $n_F = 2n_B$. What could cause such a relation between n_F and n_B? The fluctuations of the boson fields around the instanton solution $\phi_{\text{inst}}(z)$ satisfy the equations[8]

$$\frac{\partial}{\partial z} g_{\bar\alpha\beta} \frac{\partial}{\partial z} \phi^\beta_{(\epsilon)} = \epsilon^2 g_{\bar\alpha\beta} \phi^\beta(\epsilon), \quad g_{\alpha\bar\beta} = g_{\alpha\bar\beta}(\phi_{inst}, \overline{\phi_{inst}}), \tag{8}$$

while the fluctuations of the fermion fields satisfy the equations

$$\frac{\partial}{\partial \bar z} \psi^\beta_{L(\epsilon)} = \pm \epsilon \psi^\beta_{R,(\epsilon)}, \tag{9}$$

$$\frac{\partial}{\partial z} g_{\bar\alpha\beta} \psi^\beta_{R,(\epsilon)} = \pm \epsilon g_{\bar\alpha\beta} \psi^\beta_{L,(\epsilon)}. \tag{10}$$

The subscripts R and L specify the right and left components of the spinors. Clearly, for any nonzero modes ($\epsilon \neq 0$) we would have ($\epsilon \neq 0$) $\psi^\beta_{L,(\epsilon)} = \phi^\beta_{(\epsilon)}$ and $\psi^\beta_{R,(\epsilon)} = \pm(1/\epsilon)(\partial/\partial z)\phi^\beta_{(\epsilon)}$; i.e., there are two fermion modes for each boson mode.[1)] The zero modes, however, must be examined separately. As before, the one-to-one correspondence between the boson modes and the *left* fermion modes is retained: $\psi^{\beta}_{L,(0)} = \phi^{\beta}_{(\epsilon)}$, i.e., $n_{F,L} = n_B$. As for the right zero modes, we note that they cannot be of the form $(\partial/\partial\bar{z})\phi^{\beta}_{(0)}$, since the zero boson modes correspond to transitions between close instanton solutions and are analytic functions of z: $(\partial/\partial\bar{z})\phi^{\beta}_{(0)} = 0$. The number of right zero modes can be found with the help of an index theorem. The index of the Dirac operator is proportional to the Ricci tensor[9]: $\partial_{\bar{\alpha}}\Gamma^{\gamma}_{\beta\gamma} \sim R_{\bar{\alpha}\beta}$. For homogeneous Kähler manifolds this index is n_B; i.e., $n_{F,R} = n_{F,L} - n_B = 0$ and $n_F = n_{F,L} + n_{F,R} = n_B$. For Ricci-planar spaces the index is zero: $n_{F,L} = n_{F,R}$. It follows that $n_F = 2n_B$, consistent with a zero β function. Formally, the equation for right zero modes is satisfied by $\psi^{\bar{\alpha}}_{R,(0)} = g^{\bar{\alpha}\beta}(\phi^{\beta}_{(0)})^*$. In the case of homogeneous sigma models we easily see that these modes are not renormalizable. In the Ricci-planar case the functions $g^{\bar{\alpha}\beta}(\phi^{\beta}_{(0)})^*$ apparently become normalizable, but this is not an easy point to check in the absence of the explicit metric. For the same reason, the proof of the nonrenormalization theorem of Ref. 8, which ensures that there are no three-loop or higher-order corrections to the β function, now requires a more detailed analysis. The presence of zero fermion modes of both helicities in the hyper-Kähler case should not violate this theorem because of the appearance of two new generators of supertransformations which do not act on the instanton (cf. Ref. 9). We might also note that an instanton calculation of the β function is appropriate for sigma models only on compact, simply connected Kähler manifolds.

We wish to thank D. V. Alekseevskiĭ, Yu. I. Manin, A. A. Roslom, and M. A. Shifman for useful discussions.

[1)]It follows in particular that the boson determinant $\Pi_\epsilon(\epsilon^2)$ and the fermion determinant $[\Pi_\epsilon(+\epsilon)\Pi_\epsilon(-\epsilon)]$ are equal.

[1]S. Mandelstam, Nucl. Phys. **B213**, 149 (1983).
[2]L. Alvarez-Gaumé and D. Freedman, Commun. Math. Phys. **80**, 443 (1981).
[3]E. Witten, Phys. Rev. D **16**, 2991 (1977).
[4]G. W. Gibbons, in: Geometric Ideas in Physics (Russ. transl. Mir, Moscow, 1983).
[5]L. Alvarez-Gaumé, Phys. Rev. D **22**, 846 (1980).
[6]S. T. Yau, Commun. Pure Appl. Math. **31**, 339 (1978).
[7]D. N. Page, Phys. Lett. **B80**, 55 (1978).
[8]V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Preprint ITEP-188, Institute of Theoretical and Experimental Physics, 1983.
[9]A. Yu. Morozov, A. M. Perelomov, and M. A. Shifman, Preprint ITEP-14, Institute of Theoretical and Experimental Physics, 1984.

Translated by Dave Parsons
Edited by S. J. Amoretty