Motion of Brownian particles in a tilted periodic potential

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(Submitted 29 May 1984)
Pis'ma Zh. Eksp. Teor. Fiz. **40**, No. 2, 56–58 (25 July 1984)

The Langevin equation for a Brownian particle in a tilted periodic potential (TPP) with weak friction is reduced to two integral equations in energy variables. The dependence of the average velocity of the particle on the slope of the potential is found and a relationship is established between the expression obtained and the current-voltage characteristic (IVC) of a concentrated Josephson junction.

The motion of a Brownian particle in a TPP is the exact mechanical analog of fluctuations of the phase of the order parameter in a concentrated Josephson junction, which determines the IVC of the junction in a fixed range of currents and voltages. The most difficult, in terms of finding a solution, situation is the situation correspond-

 0021-3640/84/140787-05$01.00

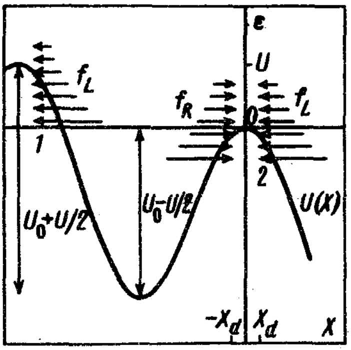

FIG. 1.

ing to the motion of a particle with a weak frictional force in a deep, relative to the thermal energy, potential relief, for which only numerical results are available.[1,2] We find below an explicit expression for the average velocity of a Brownian particle in a TPP, taking into account the quantum transparency of the barriers, and we show the relationship between this expression and the IVC of a Josephson junction.

Brownian motion is described by the Langevin equation

$$m\ddot{x} = -m\gamma\dot{x} - \partial U/\partial x + F + \eta(t), \quad (1)$$

where m and x are the mass and the coordinate of the particle, γ is the coefficient of viscous friction, $U(x)$ is the periodic potential, $U(x+L) = U(x)$, F is the slope of the potential (pulling force), and $\eta(t)$ is the stationary Gaussian fluctuation force, which is related to γ and the temperature T by the fluctuation-dissipation theorem (FDT): $\langle\eta(t)\eta(t')\rangle = 2m\gamma T\delta(t-t')$.

The potential $U(x) + Fx$ is shown in Fig. 1. The difference between the heights of barriers 1 and 2 is $U = FL$. The heights of the barriers to the left and right are $U_0 \pm U/2$, where U_0 is the height of the barrier with $F = 0$. We will assume that $U_0 \gg T$. The particles with energies at the level of the barrier will then participate in the current transport, and the Boltzmann distribution with respect to the total energy $\epsilon = p^2/2m + U(x)$, where $p \equiv m\dot{x}$ is the momentum of the particle, will be conserved deep inside the well. We are interested in the case where the particle, after overcoming the barrier, has a substantial probability of reaching the top of the next barrier. This means that the energy loss δ occurring as a result of going through one period of the potential is comparable to the temperature and is small compared to the depth of the well. We thus assume that

$$\delta \sim U \sim T \ll U_0; \quad \delta \equiv \gamma \int_0^L m\dot{x}dx = \gamma \int_0^L (-2mU(x))^{1/2}dx. \quad (2)$$

In order of magnitude we have $\delta \sim \gamma U_0/\omega$, where $\omega \sim (U_0/mL^2)^{1/2}$ is the characteristic frequency of motion in the potential $U(x)$. The condition $\delta \sim T$ means that $\gamma \sim \omega T/U_0 \ll \omega$. This is a sufficient condition for ignoring the dissipation in a region of width $\sim (T/m\omega^2)^{1/2} \ll L$ near the parabolic top of the barrier, where $U(x) \simeq m\omega^2 x^2/2$, and for taking the effect of the barrier into account only via the transmission coefficient $[1 + \exp(-2\pi\epsilon/\omega)]^{-1}$. Introducing the distribution functions $f^L(\epsilon)$ and $f^R(\epsilon)$ of the particles incident on the barrier with an energy ϵ for certain intermediate coordinates $\pm x_d$ such that $(T/m\omega^2)^{1/2} \ll x_d \ll L$, we introduce the conditions for periodicity in the form of integral equations

$$f^R(\epsilon) = \int_{-\infty}^{\infty} \frac{g(\epsilon - \epsilon' - U)}{1 + \exp(2\pi\epsilon'/\omega)} [f^R(\epsilon')\exp(2\pi\epsilon'/\omega) + f^L(\epsilon')] d\epsilon'. \tag{3}$$

$$f^L(\epsilon) = \int_{-\infty}^{\infty} \frac{g(\epsilon - \epsilon' + U)}{1 + \exp(2\pi\epsilon'/\omega)} [f^R(\epsilon') + f^L(\epsilon')\exp(2\pi\epsilon'/\omega)] d\epsilon', \tag{4}$$

where $g(\epsilon - \epsilon')$ is the Gaussian distribution of the particle in terms of the energy ϵ near one of the barriers if at the neighboring barrier the particle had an energy ϵ':$g(\epsilon - \epsilon') = (4\pi\delta T)^{-1/2}\exp[-(\epsilon - \epsilon' + \delta)^2/4\delta T]$.

This kernel is completely determined by the fact that the loss of energy in the period is equal to δ, while the energy dissipation in accordance with the FDT, is $2(\delta T)^{1/2}$. Equation (3) shows, for example, that the function $f^R(\epsilon)$ at the barrier 2 is formed from particles passing through the barrier 1 [the term with $f^R(\epsilon')$ on the right side of (3)] and from particles reflected from barrier 1 [the term with $f^L(\epsilon')$, see Fig. 1]. The shift in the argument by $\pm U$ in the kernels of Eqs. (3) and (4) corresponds to different origins for measuring the energy at barriers 1 and 2.

The average velocity of the particle $\langle \dot{x} \rangle$ is determined in terms of the functions $f^R(\epsilon)$ and $f^L(\epsilon)$ by the expression

$$\langle \dot{x} \rangle = L \int_{-\infty}^{\infty} \frac{f^R(\epsilon) - f^L(\epsilon)}{1 + \exp(-2\pi\epsilon/\omega)} d\epsilon. \tag{5}$$

We will write the boundary conditions for $f^R(\epsilon)$ and $f^L(\epsilon)$ under the assumption that most of the time the particle is at the bottom of the potential well. Working from the standard normalization in the variables p and x to a single particle in a calculation for a period, we find

$$f^{R,L} = \frac{\Omega}{2\pi T} \exp(-(U_0 \mp U/2 + \epsilon)/T); \quad -\epsilon \gg T, \tag{6}$$

where Ω is the oscillation frequency of the particle at the bottom of the well.

A substitution of the form

$$\varphi(\lambda) = \int_{-\infty}^{\infty} \frac{f(\epsilon)\exp(i\lambda\epsilon/T)}{1 + \exp(2\pi\epsilon/\omega)} d\epsilon \tag{7}$$

reduces Eqs. (3) and (4) to finite-difference equations

$$\begin{aligned} &\varphi^R(\lambda) + \varphi^R(\lambda - 2\pi iT/\omega) = g_-(\lambda)[\varphi^R(\lambda - 2\pi iT/\omega) + \varphi^L(\lambda)], \\ &\varphi^L(\lambda) + \varphi^L(\lambda - 2\pi iT/\omega) = g_+(\lambda)[\varphi^R(\lambda) + \varphi^L(\lambda - 2\pi iT/\omega)], \end{aligned} \tag{8}$$

where $g_\pm(\lambda) = \exp[-\delta\lambda^2/T - i\lambda(\delta \pm U)/T]$. Solving these equations for $\phi^{R,L}(\lambda - 2\pi iT/\omega)$, we find for the function $\phi(\lambda) = \phi^R(\lambda) - \phi^L(\lambda)$

$$\varphi(\lambda - 2\pi iT/\omega) = -G(\lambda)\varphi(\lambda), \tag{9}$$

where

$$G(\lambda) \equiv [1 - g_+(\lambda)g_-(\lambda)]/(1 - g_+(\lambda))(1 - g_-(\lambda)).$$

The solution of Eq. (9), with the boundary condition derived from (6) and (7)

$$\varphi(\lambda) = -\frac{i\Omega\omega}{\pi}\frac{\sinh(U/2T)\exp(-U_0/T)}{\lambda + i}; \qquad |\lambda + i| \ll 1,$$

is

$$\varphi(\lambda) = -\frac{i\Omega\omega}{2\pi T}\frac{\sinh(U/2T)\psi(\lambda)\exp(-U_0/T)}{\sinh[\omega(\lambda + i)/2T]\psi(-i)}, \tag{10}$$

where

$$\psi(\lambda) = \exp\left\{\frac{\omega}{T}\int_{-\infty}^{\infty}\frac{d\lambda'}{4\pi i}\frac{\ln G(\lambda')}{\tanh[\omega(\lambda' - \lambda)/2T]}\right\}. \tag{11}$$

Comparison of (5) to (7) gives $\langle\dot{x}\rangle = L\phi(-2\pi iT/\omega)$. Using (10) and (11), after a series of transformations, we find

$$\langle\dot{x}\rangle = \frac{\Omega\omega L\sinh(U/2T)F^2(2\delta, 1, 1)\exp(-U_0/T)}{2\pi T\sin(\omega/2T)F(\delta, \alpha, \alpha)F(\delta, \beta, \beta)F(\delta, \alpha, \beta)F(\delta, \beta, \alpha)}, \tag{12}$$

where $\alpha = 1 + U/\delta$, $\beta = 1 - U/\delta$ and the function $F(\delta,\alpha,\beta)$ is defined by the expression

$$\ln F(\delta, \alpha, \beta) = \frac{\omega\sin(\alpha\omega/2T)}{T}\int_{-\infty}^{\infty}\frac{d\lambda}{4\pi}\Phi(\delta, \alpha, \beta, \lambda);$$

$$\Phi(\delta, \alpha, \beta, \lambda) = \ln\left[1 - \exp\left(-\frac{\delta}{T}\left(\lambda^2 + \frac{\beta^2}{4}\right)\right)\right]\Big/\left[\cosh(\omega\lambda/T) - \cos(\alpha\omega/2T)\right].$$

Expression (12) is valid up to the separation point, $U = \delta$, near which $\langle\dot{x}\rangle \sim (1 - U/\delta)^{-1}$. The continuation of the IVC through this point will be described separately.

To find the IVC of a Josephson junction it is necessary to take into account the fact that the average voltage V is comparable to $\langle\dot{x}\rangle$, while the slope of the potential U is comparable to the current I passing through the junction. These quantities are described by the relations

$$V = \pi\hbar \langle \dot{x} \rangle / eL; \quad U = \pi\hbar \dot{I}/e.$$

The parameters U_0,Ω,ω, and δ entering into (12) are related to the critical current I_c through the capacitance C and resistance R of the junction by the relations

$$U_0 = \hbar I_c/e; \quad \Omega = \omega = (2eI_c/\hbar C)^{1/2}; \quad \delta = \frac{2}{R}^{3/2}\left(\frac{\hbar}{e}\right)^{3/2}\left(\frac{I_c}{C}\right)^{1/2},$$

where e is the electron charge. For clarity the Planck's constant is explicitly written everywhere.

[1]Ju. Kurkijarvi and V. Ambegaokar, Phys. Lett. A **31**, 314 (1970).
[2]R. F. Voss, J. Low Temp. Phys. **42**, 151 (1981).

Translated by M. E. Alferieff
Edited by S. J. Amoretty