Harmonic superspace: key to $N=2$ supersymmetry theories

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The concept of a harmonic $N=2$ superspace with additional coordinates related to the SU(2)/U(1) sphere is introduced. This concept leads to an adequate geometric description of the $N=2$ theories of matter, the Yang-Mills theory, and supergravity in terms of superfields without coupling. A new effect has been discovered: the unboundedness of the number of gauge and auxiliary degrees of freedom.

1. So far, it has not been found possible to construct expanded supersymmetry theories (Yang-Mills theories, theories of supergravity, or theories of matter) by an explicitly covariant geometric approach in terms of superfields not subject to constraints. Partial success has been achieved in a nongeometric approach to the Abelian Yang-Mills theory[1] and its generalizations (in the form of a recursive procedure) to the non-Abelian case.[2] The need for such a construction is urgent in many regards, primarily in connection with the remarkable property of finiteness of several supersymmetry models of field theory.

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2. We propose such a construction in the present letter. It is based on the introduction of harmonic variable $u_i^{\pm}$, the coordinates of the SU(2)/U(1) sphere, in addition to the ordinary even coordinates. The variables $u_i^{\pm}$ are dyads; they are assigned indices of two types, an SU(2) index (i) and a U(1) index ($\pm$), and they form a bridge between these groups. Making use of the normalization condition

$$u^{+i}u_i^- = 1, \tag{1}$$

we can reversibly convert the ordinary spinor coordinates $\theta^i_\alpha, \bar{\theta}^i_{\dot{\alpha}}$ with the indices i of the SU(2) group into spinor coordinates $\theta^{\pm}_\alpha, \bar{\theta}^{\pm}_{\dot{\alpha}}$ with the indices $\pm$ of the U(1) group:

$$\theta^{\pm}_\alpha = \theta^i_\alpha u_i^{\pm}, \qquad \bar{\theta}^{\pm}_{\dot{\alpha}} = \bar{\theta}^i_{\dot{\alpha}} u_i^{\pm} \tag{2}$$

$$\theta^i_\alpha = u^{+i}\theta^-_\alpha - u^{-i}\theta^+_\alpha, \quad \bar{\theta}^i_{\dot{\alpha}} = u^{+i}\bar{\theta}^-_{\dot{\alpha}} - u^{-i}\bar{\theta}^+_{\dot{\alpha}}. \tag{3}$$

A key point here is that the subspace

$$\{\zeta_A = (x^m_A, \theta^+_\alpha, \bar{\theta}^+_{\dot{\alpha}}), u^{\pm}\}, \qquad x^m_A = x^m - 2i\theta^{(i}\sigma^m\bar{\theta}^{j)}u_k^+ u_j^- \tag{4}$$

(not including the coordinates $\theta^-_\alpha, \bar{\theta}^-_{\dot{\alpha}}$) is closed under the transformations of the $N=2$ supersymmetry:

$$\delta x^m_A = -2i(\epsilon^k\sigma^m\bar{\theta}^+ + \theta^+\sigma^m\bar{\epsilon}^k)u_k^-, \tag{5}$$

$$\delta\theta^+_\alpha = \epsilon^i_\alpha u_i^+, \quad \delta\bar{\theta}^+_{\dot{\alpha}} = \bar{\epsilon}^i_{\dot{\alpha}} u_i^+, \quad \delta u^{\pm} = 0.$$

We call (4) the "analytic superspace of $N=2$ supersymmetry," and the superfields $\Phi^{(q)}(\zeta_A, u)$ defined on it are "analytic." A superfield $\Phi^{(q)}(\zeta_A u)$ as a whole is a representation of U(1) with a charge q. Its components have charges ranging from q to $q-4$, depending on the number θ^+, θ^+ in the given term of the expansion. Each component is in turn a function of the new coordinates $u_i^{\pm}$ which permits an infinite expansion of the type

$$F^{(q)}(x_A, u^{\pm}) = \sum_{n=0}^{\infty} f^{(i_1 \ldots i_{n+q} j_1 \ldots j_n)}(x_A)\, u^+_{i_1} \ldots u^+_{i_{n+q}} u^-_{j_1} \ldots u^-_{j_n}. \tag{6}$$

In the expansion of $\Phi^{(q)}$ the charges of the U(1) group carry only harmonic variables, the dyads $u_i^{\pm}$ (and the spinor coordinates θ^+, θ^+, which contain them). Remarkably, the components fields $f^{(i_1 \ldots j_n)}(x_A)$ are representations of the SU(2) group and scalars under U(1)! The reason is that expansion (6) is actually a harmonic expansion on a homogeneous space, the SU(2)/U(1) sphere.[3] Analysis shows that of the two quantum numbers of the $N=2$ supersymmetry the first—the superspin—is identical for all terms of the expansion and vanishes for scalar analytic superfields with the U(1) charge q, while the second—the superisospin I—takes on the values

$$\Phi^{(q)}(\zeta_A, u): \quad I = \left|\frac{q}{2} - 1\right| + n, \quad n = 0, 1, 2, \ldots \tag{7}$$

We also wish to emphasize that one could define a covariant harmonic derivative D^{++} of such a nature that in acting on an analytic superfield it leaves this field analytic and is compatible with normalization (1):

$$D^{++}\Phi^{(q)} = \left(u^{+i}\frac{\partial}{\partial u^{-i}} - 2i\theta^{+}\sigma^{m}\bar{\theta}^{+}\frac{\partial}{\partial x_A^m} \right)\Phi^{(q)} . \quad (8)$$

We have thus been able to generalize the Grassmann analyticity[4] in such a way that the SU(2) symmetry is conserved! The analytic variables from Ref. 4 can be found from (2) by making the particular choices $u^{+1} = -(1/\sqrt{2})$, $u^{+2} = -(i/\sqrt{2})$. We have already found a coordinate of the type x_A^m from (4) (see the bases in Ref. 5). It is important to note that we have the operation$\overset{*}{}$ which is the product of a complex conjugation and the operation*,

$$* \quad u^{\pm i} \to \pm u^{\mp i} , \quad (9)$$

which maps an analytic subspace into itself. The operation$\overset{*}{}$ can be used to determine analytic real superfields. In constructing an action we need an integral over the analytic superspace. An integral over the harmonic coordinates is defined by the following rules:

$$\int du \cdot 1 = 1, \int du\, u^{(+i_1} \ldots u^{+i_p} u^{-j_1} \ldots u^{-j_q)} = 0, \quad p + q > 0. \quad (10)$$

The volume element is $d\zeta_A^{(-4)} du = d^4 x_A d^2\theta^{+} d^2\bar{\theta}^{+} du$.

3. We turn now to the construction of $N = 2$ supersymmetry theories. We begin with the Fayet-Sohnius matter hypermultiplet.[6,7] This hypermultiplet has a superspin 0 and a superisospin 1/2. According to (7), the simplest superfield which contains it is the analytic superfield q^+. For it, the action is written in the form[2)]

$$S = \int d\zeta_A^{(-4)} du \left[\overset{*}{\bar{q}}{}^{+} D^{++} q^{+} + \frac{\lambda}{2} (\overset{*}{\bar{q}}{}^{+})^2 (q^{+})^2 \right] + \text{H.a.} \quad (11)$$

and implies the equations of motion

$$D^{++} q^{+} = - \overset{*}{\bar{q}}{}^{+} (q^{+})^2 . \quad (12)$$

In studying the superfield q^+, we find a fundamentally new phenomenon. According to (7), q^+ contains an infinite number of auxiliary fields, which form supermultiplets with superspin 0 and infinitely increasing superisospins 1/2,3/2. . . . It follows from equations of motion (12) that all fields with a superisospin exceeding 1/2 are zero in the free case ($\lambda = 0$) and can be expressed in terms of fields with a superisospin 1/2 at $\lambda \neq 0$.

An analogous situation arises for a hypermultiplet of another type.[8] It has a superisospin 1 and a superspin 0 and is described by an analytic superfield ω which contains, according to (7), superspin 1, 2,. . . . The action for ω in the free case is written

$$S_0 = \int d\zeta_A^{(-4)} du\, D^{++} \omega\, D^{++} \omega , \quad (13)$$

and analysis of the equations of motion $(D^{++})^2 \omega = 0$ shows that an infinite set of auxiliary supermultiplets with superisospins $\geqslant 2$ on the mass shell vanishes. It is easy to construct an interaction, e.g., of the type

$$S_{int+0} = \int d\zeta_A^{(-4)} du\, g^{ab}(\kappa\omega) D^{++} \omega_a D^{++} \omega_b , \quad (14)$$

where $g^{ab}(\kappa\omega)$ is the "metric," and κ is the coupling constant. Howe *et al.*[8] actually use the coupling (in our notation) $(D^{++})^3\,\omega = 0$, which is compatible only with free equations of motion.

4. An $N = 2$ Yang-Mills theory has previously been known in component form,[9,6] in the form of a superfield theory with coupling[2)] (Ref. 10), and in a nongeometric description with prepotentials of higher dimensionality.[1,2] The Yang-Mills $N = 2$ supermultiplet contains the vector field $A_a(x)$, the scalar fields $M(x)$ and $N(x)$, the triplet $D_{ij}(x)$, and the Majorana isodoublet $\Psi^i_\alpha(x), \bar{\Psi}_{\dot{\alpha}i}(x)$. Its superspin and superisospin are both zero. A supermultiplet of this sort could be described by a superfield V^{++} (superspin 0, superisospins 0, 1, ...). The extraneous superisospins are "decontaminated" by gauge transformations:

$$(V^{++})' = \frac{1}{ig}\, e^{i\omega}(D^{++} + igV^{++})\, e^{-i\omega}, \qquad \begin{aligned} V^{++} &= V_i^{++}\, T_i, \\ \omega &= \omega_i T_i, \end{aligned} \tag{15}$$

where the T_i are the generators of the internal-symmetry group. Transformation (15) literally copies a transformation from the ordinary Yang-Mills theory ($N = 0$): The derivative $\partial/\partial x^m$ is replaced by the derivative D^{++}, the V_m coupling is replaced by a V^{++} coupling, and the gauge function $\omega(x)$ is replaced by the gauge analytic superfield $\omega(\zeta_A, u)$. A remarkable new phenomenon is that we are now dealing with an infinite number of gauge degrees of freedom. We will postpone a discussion of the action to a more detailed paper; at this point, we simply note that it is also a simple matter to incorporate an interaction with material fields: It is necessary to lengthen the harmonic derivative, $D^{++} \to D^{++} + igV^{++}$, in (11), (13), and (14).

5. We now consider the $N = 2$ Einstein supergravity. A fundamental gauge group is chosen by requiring conservation of the analytic representations, in the manner in which it is chosen from the requirement of the conservation of chirality in the $N = 1$ case.[13] The analytic coordinates must be supplemented in this case by the coordinate of the central charge, x^5_A (in order to describe the graviphoton). The gauge group is realized as the group of coordinate-independent representations in $\{\zeta^M_A, x^5_A, u^\pm\}$ which leave invariant the subspace $\{\zeta^M_A, u^\pm\}$:

$$\delta\zeta^M_A = \lambda^M(\zeta_A, u), \quad \delta x^5_A = \lambda^5(\zeta_A, u). \tag{16}$$

As before, the harmonic variables $u_i^\pm$ undergo only global SU(2) and U(1) rotations. The basic geometric quantities of the theory are the $++$ components of the analytic frame of reference

$$V^{++M} = V^{++M}(\zeta_A, u), \qquad V^{++5} = V^{++5}(\zeta_A, u) \tag{17}$$

with the transformation laws

$$\delta V^{++M,\,5} = V^{++M,\,5\prime}(\zeta'_A, u) - V^{++M,\,5}(\zeta_A, u) = D^{++}\lambda^{M,\,5}(\zeta_A, u), \tag{18}$$

where

$$D^{++} = u^{+i}\frac{\partial}{\partial u^{-i}} + V^{++M}(\zeta_A, u)\frac{\partial}{\partial\zeta^M_A} + V^{++5}(\zeta_A, u)\frac{\partial}{\partial x^5_A} \tag{19}$$

is the covariant version of derivative (8). It can be shown that in the Wess-Zumino gauge the $V^{++M,5}$ component composition is precisely the same as the composition of the multiplet of the $N=2$ Einstein supergravity (in its original version[14]). A differential geometry can be constructed by analogy with the $N=2$ Yang-Mills case. All the constraints are again solved in terms of prepotentials $V^{++M,5}(\zeta_A, u)$.

6. In summary, all the expanded $N=2$ supersymmetry theories allow an explicitly covariant formulation without the imposition of constraints of any sort. While the $N=1$ supersymmetry requires complexification, a "harmonization" arises in the $N=2$ case. A fundamentally new phenomenon which emerges in this case is the infinite number of gauge and/or auxiliary degrees of freedom. It appears to us that general phenomenon can explain the sources of the known theorems stating that it is impossible to find a finite number of auxiliary fields in $N=4$ theories, which await development.

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[1)]The self-action may also include other terms. It is easily generalized to the case of several hypermultiplets.
[2)]Roslyĭ[11] uses variables of the type u_i in analyzing the constraints of an $N=2$ Yang-Mills theory in the spirit of Ward's paper.[12]

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