Excitation of random waves in a nonlinear chain

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Numerical simulations are reported on the randomization of a wave field excited by a periodic point source in a matched line which models a semi-infinite nonlinear chain. If the source amplitude is sufficiently high, almost perfectly stochastic waves are excited.

Random oscillations in nonlinear systems with a small number of degrees of freedom have now been studied in some detail.[1] The ideas that have been advanced can also be applied to certain problems involving the randomization of nonlinear wave fields, but the situations that are being discussed here (random steady-state waves,[2] waves which are random only in space[3] or only in time,[1,4] and random regimes in distributed bounded systems[5,6]) are not fundamentally different from lumped models. In the present letter we describe the randomization of a wave field in a system which models an infinite number of degrees of freedom.

We consider a semi-infinite nonlinear medium at whose boundary a periodic external source is operating. A problem of this type arises, for example, in studies of the effect of intense, steady-state electromagnetic waves on a plasma and in studies of the propagation of laser beams through crystals. As a specific model for the numerical simulation we adopt the nonlinear chain described by the Hamiltonian

$$H(p_i, q_i, t) = q_1 A\cos\omega_0 t + \frac{1}{2}\sum_{i\geqslant 1}\left[p_i^2 + q_i^2 + \frac{1}{2}q_i^4 + D(q_i^2 - q_{i-1}^2)\right] \qquad (1)$$

with the boundary condition $q_0 = q_1$. The first term in this Hamiltonian corresponds to an external force of frequency ω_0 and amplitude A, which is applied at the boundary $i = 1$. Hamiltonian (1) is a discrete analog of the ϕ^4 Hamiltonian in field theory, which leads to the nonlinear Klein-Gordon equation. The spectrum of linear waves in chain (1) is continuous:

$$\omega^2(k) = 1 + 2D(1 - \cos k), \quad 0 \leqslant k \leqslant \pi.$$

For the simulation of the semi-infinite medium in the calculations, we add to a bounded segment of the chain $(1 \leqslant i \leqslant N)$ several elements in which we introduce a damping. As a result, the waves which propagate in the region $i > N$ are damped and are essentially unreflected (the linear standing-wave ratio is less than 3%). This method clearly shows that "from the standpoint of the source" a conservative infinite medium may be regarded as dissipative. In particular, the steady-state regime does not depend on the initial conditions.

The numerical simulations were carried out for $N = 150$, $D = 20$, and $\omega_0 = 2\pi/3$ at various amplitudes of the external force, A. At small values of A $(A \lesssim 30)$ a traveling periodic wave is established in the chain. At $30 \lesssim A \lesssim 70$ we see a regime with a slight tendency toward a stochastic situation. A periodic wave is excited near the $i = 1$

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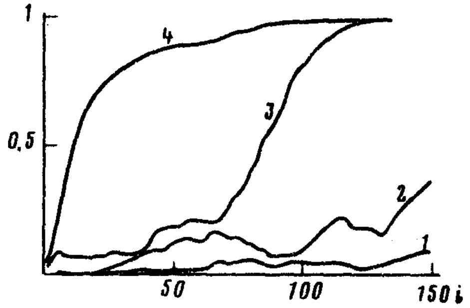

FIG. 1. Distribution along the chain of the fraction of the power in the oscillation spectrum of the particles. 1—$A = 25$; 2—$A = 40$; 3—$A = 65$; 4—$A = 75$.

boundary. In the course of the propagation, the periodicity is disrupted, and a random component arises. Also during the propagation, the interaction of the regular wave with the noise (the random component) causes the noise intensity to increase.[7] It can be expected that far from the source ($i \to \infty$) the wave field will become totally random: All the energy will be transferred to the random component. At $A \gtrsim 70$ we see a different situation. Here the randomization of the waves occurs even in the near field of the source. As a result, an almost perfectly randomized wave field forms at a short distance from the source. The situation is illustrated by Fig. 1, which shows the distribution along the system of the fraction of the total power in the continuous spectrum.

Let us briefly discuss the statistical properties of the random waves excited in the chain. Figure 2 shows the distribution function of the chain at the point $i = 140$ for $A = 75$. It is nearly a normal distribution. Figure 3 shows the power spectrum of the waves at the same point. This spectrum is continuous and roughly constant over the interval $\omega_0 < \omega < \omega_m$, where ω_0 is again the frequency of the external force, and $\omega_m = \sqrt{1 + 4D} = 9$ is the maximum frequency of the linear waves. These results may be interpreted as follows: The energy of the source is distributed uniformly among the

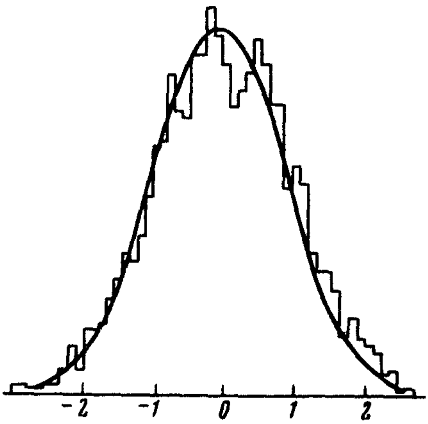

FIG. 2. Histogram of the displacements of the 140th element of the chain for $A = 75$. The curve is a normal distribution with the same mean and the same variance.

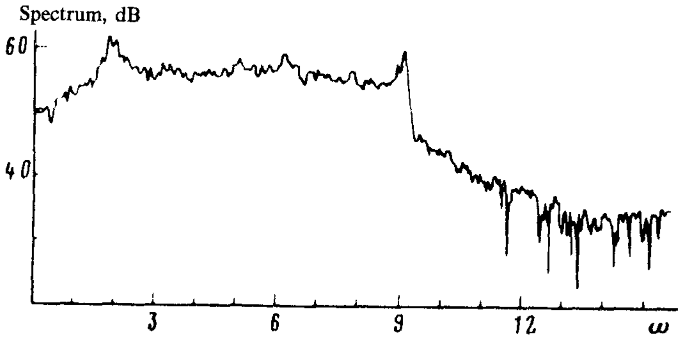

FIG. 3. Power spectrum of the oscillations of the 140th element of the chain for $A = 75$.

wave components lying in the transparency band of the chain, and these waves may be regarded as independent.

Let us summarize the results of this study. For the particular case of a chain of nonlinear oscillators it has been shown that excitation of an infinite nonlinear medium by a point source can lead to the excitation of random waves. Two routes for the transition of the waves to a stochastic state can be identified. The first is a randomization in the course of the propagation. This type of randomization apparently does not operate if the nonlinear medium is described by integrable equations or if the waves decay quite rapidly with distance from the source (because of damping or because of a divergence in a $2D$ or $3D$ medium). The second and more effective route to randomization, but one which is seen at high amplitudes, is a randomization of the waves directly in the near field around the source. In this case waves, whose spectrum contains a continuous component, propagate in the medium. In the course of the propagation the intensity of the noise component rises because of the nonlinear interaction, and at large distances from the source we see an essentially completely random noise field.

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