Fine structure in toroidal plasma configurations

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The applicability of the near-equilibrium method is demonstrated for a dissipative plasma. Under a sufficient condition derived here, an equilibrium toroidal plasma configuration should have a fine structure consisting of a set of magnetic islands. For a tokamak, this condition holds in the region $q(\rho) < 1$.

According to the present understanding, the equilibrium toroidal plasma configurations (tokamaks, stellarators, and stabilized pinches) are systems of nested magnetic surfaces of comparatively simple topology. The appearance of an additional structure—magnetic islands—is attributed to either the presence of resonant perturbations in the magnet system or the onset of tearing-mode instabilities. In this letter we formulate a sufficient condition under which a toroidal configuration should spontaneously convert, by virtue of the equilibrium conditions alone, into a new near-equilibrium state consisting of a set of magnetic islands. The conventional understanding of the structure of the configuration, on which the transport theory, in particular, is based, would then become incorrect.

To simplify the analysis, we consider the condition for the appearance of a near equilibrium for a straight plasma column of circular cross section. We seek a configuration with helical symmetry characterized by the parameter $\kappa = n/mR$, where m and n are wave numbers, and R is the radius of the equivalent torus. The equilibrium equation for the helical-flux function Ψ^*, which determines the shape of the magnetic surfaces, is[1]

$$\Delta^*\Psi^* = J(\rho, \Psi^*), \tag{1}$$

where Δ^* is the analog of the Laplacian for helical symmetry, and the functional dependence of the right side, $J(\rho, \Psi^*)$, on Ψ^* is determined by the current-density distribution, $i(\rho, \Psi^*)$, and by the pressure distribution, $p(\Psi^*)$, along the magnetic surfaces.

We assume that the initial configuration is described by the flux function

$$\Psi^* = \Psi_0^*(\rho) = -\int_0^\rho (B_\omega - \kappa\, \rho B_s)\, d\rho, \tag{2}$$

where B_ω and B_s are the components of the magnetic field. We assume that there is a resonant surface at the point $\rho = \rho_s$, and on this surface the condition $d\Psi_0^*/d\rho = 0$ holds. For a linear helical perturbation $\Psi^* = \Psi_0^* + Y(\rho)\cos m\theta$, we find the following equation, which holds everywhere except in an infinitesimally small neighborhood of the resonant surface:

$$\Delta^* Y(\rho)\cos m\theta = \left.\frac{\partial J(\rho, \Psi^*)}{\partial \Psi^*}\right|_{\Psi^* = \Psi_0^*} Y(\rho)\cos m\theta + J^1, \tag{3}$$

where the increment $J^1 = J^1(\rho, \Psi^*)$ stems from a possible perturbation of the dependence of the current and the pressure on Ψ^*.

Because of the term J^1 in (3), it may seem that this equation for an MHD equilibrium is insufficient for analyzing the possible appearance in the initial configuration of a near equilibrium related in an evolutionary way. In the linear approximation, however, we have $J^1(\rho, \Psi^*) = J^1(\rho, \Psi_0^*)$, and since this relation is incompatible with the angular dependence of the other terms in Eq. (3), we must set $J^1 = 0$. Near a resonant surface that splits into a magnetic island, on the other hand, the only assumption is the continuity of the current and the pressure, which is maintained in any case over times shorter than those required for the formation of the current and pressure profiles.

Consequently, the equilibrium equation by itself is sufficient for analyzing the appearance of a near equilibrium. Regardless of Ohm's law and the transport phenomena, which are known to be anomalous, an invariant of the transition to the near helical equilibrium is the functional dependence $J(\rho, \Psi^*)$.

With $J^1 = 0$, Eq. (3) is equivalent to the Euler equation for kink perturbations, which is well known in the theory of hydromagnetic stability.[2] In the most important case, $B_s \gg B_\omega$, this equation is

$$\frac{d}{d\rho}\rho\frac{dY}{d\rho} - \frac{m^2}{\rho}Y = \frac{Y}{\mu - n/m}\left[\frac{4\pi R}{cB_s}j_s' + \frac{8\pi\kappa^2 p'(\rho)R^2}{(\mu - n/m)B_s^2}\right], \tag{4}$$

where $\mu = RB_\omega/\rho B_s$ is the rotational transform. For a continuous current density near an island, Eq. (4) must be solved under the condition that the internal solution Y_i and the external solution Y_e and their derivatives are to be joined at the resonant surface. Near this surface we have the expansion ($x = \rho - \rho_s$)

$$Y_{i,e} = \left|\frac{x}{\rho_s}\right|^{\nu} - \frac{4\pi}{c}\frac{j_s' x}{2\mu'\rho_s\nu}\left(\left|\frac{x}{\rho_s}\right|^{\nu} - \left|\frac{x}{\rho_s}\right|^{-\nu}\right) + \alpha_{i,e}\left|\frac{x}{\rho_s}\right|^{1-\nu}, \tag{5}$$

where

$$\nu = \frac{1}{2}\left[1 - (1 - 4D)^{1/2}\right], \quad D = -\frac{8\pi\rho_s p'}{B_s^2}\left(\frac{\mu}{\mu'\rho_s}\right)^2, \tag{6}$$

and the coefficients $\alpha_{i,e}$ are determined by the boundary conditions. The difference $\alpha_e - \alpha_i$ is equal to the value of $\rho_s \Delta'$ for the tearing mode, which is also described by Eq. (14) if we omit the term with the pressure.

The first term in (5) is the dominant one, so that a near equilibrium arises in the linear approximation with $D = 0$. If $D < 0$, the system has no such new equilibrium, while at $D > 0$ the configuration must change, with the result that a magnetic island forms. The island half-width w can be estimated from (5) to be $w/\rho_s = -2D/\rho_s\Delta'$, in agreement with a quasilinear calculation.

The analog of D in an arbitrary toroidal configuration can easily be determined by putting the Mercier criterion in the form $1/4 - D > 0$, by dividing all the terms in it by the square of the shear S. In particular, for a tokamak (of circular cross section) we have

$$D = \frac{8\pi\rho_s p'}{B_s^2 S^2}(q^2 - 1), \quad q = 1/\mu, \tag{7}$$

so that in the plasma region, in which the condition $q^2(\rho) < 1$ holds, all the magnetic surfaces must split into magnetic islands. The same is true of stabilized pinches, for which the condition for a transition to a near equilibrium ($D > 0$) always holds.

It is interesting to note that in the case of an unstable tearing mode ($\Delta' > 0$), but with $D < 0$ (a situation that frequently arises, e.g., for the $m = 2$ mode in a tokamak) the plasma is stable with respect to small perturbations but unstable with respect to finite-amplitude perturbations on the resonant surface, $Y > \rho B_s \mu'(D/\rho_s\Delta')^2/R$. This situation may explain the unexpected triggering of a kink perturbation which is observed during disruptions in tokamaks.

The independence of the near-equilibrium theory for many anomalous effects in toroidal systems makes its conclusions applicable to a real plasma. This calculation for one separate mode demonstrates that the magnetic configuration has a fine structure under the conditions formulated above, but this calculation is not sufficient for determining how this structure affects the transport processes. In future work we intend to study the collective effects of many modes.

[1]L. S. Solov'ev and V. D. Shafranov, in: Voprosy teorii plazmy (ed. M. A. Leontovich), Atomizdat, Moscow, Vol. 5, 1967, p. 58 (Reviews of Plasma Physics, Vol. 5, Consultants Bureau, New York, 1967). Atomizdat, Moscow, 1967, p. 58.

[2]B. B. Kadomtsev, in: Voprosy teorii plazmy (ed. M. A. Leontovich), Vol. 2, Atomizdat, Moscow, 1963, p. 157 (Reviews of Plasma Physics, Vol. 2, Consultants Bureau, New York, 1966).

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