Generalized Lie superalgebras and a supergravity with a positive cosmological constant

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A new law for forming the Hermitian conjugation makes it possible to construct a Hermitian action for a supergravity with a positive cosmological constant Λ. This modified conjugation leads to generalized ($Z_2 \times Z_2$-gauge) Lie superalgebras that correspond to a supergravity with $\Lambda > 0$.

In a supergravity,[1] in contrast with a gravity, the cosmological constant satisfies the restriction[2–4] $\Lambda \leqslant 0$ (at supersymmetry stationary points). The physical meaning of this restriction is an additional cosmological attraction. The reason is that with $\Lambda \neq 0$ supersymmetry requires[2–4] the introduction of mass-like terms for the gravitino with a mass parameter that satisfies the relation $\Lambda = -3m^2$. As a result, at $\Lambda > 0$ the parameter m turns out to be purely imaginary, and the mass term of the gravitino,

$$S^{3/2}_{m} = m\epsilon^{\nu\mu\rho\sigma} \int d^4x \,(h_\nu{}^{\alpha}{}_{\dot\beta} h_\mu{}^{\gamma\dot\beta} \psi_{\rho\alpha}\psi_{\sigma\gamma} - h_{\nu\gamma}{}^{\dot\beta} h_\mu{}^{\gamma\dot\delta}\, \phi_{\rho\dot\beta}\phi_{\sigma\dot\delta}) \tag{1}$$

($h_{\nu\alpha\dot\beta}$ is a tetrad; $\psi_{\nu\alpha}$ and $\phi_{\nu\dot\beta}$ are the conjugate fields of the gravitino; $\nu, \mu, \rho, \ldots = 0$–3; $\alpha, \beta, \ldots = 1, 2$; $\dot\alpha, \dot\beta, \ldots = 1, 2$), becomes anti-Hemitian in terms of the standard definition of conjugation: $\psi^+_{\rho\alpha} = \phi_{\rho\dot\alpha}$, $\phi^+_{\rho\dot\alpha} = \psi_{\rho\alpha}$, $h^+_{\nu\alpha\dot\beta} = h_{\nu\beta\dot\alpha}$.

In many regards, however, it is the opposite case $\Lambda > 0$ (the de Sitter case) that is of interest; this case corresponds physically to an additional repulsion. We would like to point out in this connection that if the conjugation law is modified to

$$\psi^*_{\rho\alpha} = i\phi_{\rho\dot\alpha}, \quad \phi^*_{\rho\dot\beta} = -i\,\psi_{\rho\beta}, \quad h^*_{\nu\alpha\dot\beta} = h_{\nu\beta\dot\alpha}, \tag{2}$$

it is found that both the mass term (1) and the entire action of an $N = 1$ supergravity[1,4] are self-adjoint specifically in the case $\Lambda > 0$ (the case $N > 1$ can be treated in a completely analogous way). Although we are assuming that the operation $*$ in (2) is antilinear ($i^* = -i$) and reverses the order of the operators, this operation cannot be considered a Hermitian conjugation, since the following conditions hold, as is easily shown:

$$(F^*)^* = -F, \quad (B^*)^* = B, \tag{3}$$

where F and B are arbitrary fermion and boson fields. Nevertheless, by using the Klein operator $K = e^{i\pi n}_F$ (the operator n_F represents the fermion number of the particles), which has the properties

$$KB = BK, \quad KF = -FK, \quad K^2 = 1, \quad K = K^+, \tag{4}$$

we can associate a Hermitian conjugation (involution) $+$ with any operation of the type in (3) by setting

$$F^+ = i\,KF^*, \quad B^+ = B^*. \tag{5}$$

As a result, since we have $B^+ = B^*$ on bosons, the action of an $N = 1$ supergravity with $\Lambda > 0$ turns out to be Hermitian if

$$\psi^+_{\rho\alpha} = -K\phi_{\rho\dot\alpha}, \quad \phi^+_{\rho\dot\alpha} = K\psi_{\rho\alpha}, \quad (h_{\nu\alpha\dot\beta})^+ = h_{\nu\beta\dot\alpha}. \tag{6}$$

Unfortunately, although the conjugation law proposed here does make the action of a supergravity with $\Lambda > 0$ Hermitian, it can be used in practice only in a space of states with an indefinite metric in the fermion sector (and at $N > 1$ in the sector of spin-1 fields having the incorrect sign for the kinetic term), so that it becomes difficult to physically interpret the theory. Nevertheless, since the action of a supergravity with $\Lambda > 0$ remains invariant under local supertransformations with parameters ϵ_α and $\xi_{\dot\alpha}$ satisfying Hermitian conditions of the type in (6), we are quite interested in the corresponding algebra of the supersymmetry. Quite unexpectedly, the resolution of this question (described below) goes beyond the scope of ordinary Lie superalgebras.

The Lie superalgebra osp (N, 4; $\mathbb{C}$), which corresponds to a complexification of the expanded anti-de Sitter superalgebra osp (N, 4; $\mathbb{R}$), can be specified by the generators $L_{\alpha\beta}, N_{\dot\alpha\dot\beta}, \mathscr{P}_{\alpha\dot\beta}, T^{ij}, Q^i_\alpha, R^j_{\dot\beta}$ $(i, j, \ldots = 1 \ldots, N)$ with the product law

$$[\mathscr{P}_{\alpha\dot\beta}, \mathscr{P}_{\gamma\dot\delta}] = 2\lambda^2 (\epsilon_{\alpha\gamma} N_{\dot\beta\dot\delta} + \epsilon_{\dot\beta\dot\delta} L_{\alpha\gamma}); \tag{7}$$

$$[\mathscr{P}_{\alpha\dot\beta}, Q^i_\gamma] = \lambda\epsilon_{\alpha\gamma} R^i_{\dot\beta}, \qquad [\mathscr{P}_{\alpha\dot\beta}, R^i_{\dot\delta}] = \lambda\epsilon_{\dot\beta\dot\delta} Q^i_\alpha; \tag{8}$$

$$\{Q^i_\alpha, Q^j_\beta\} = \lambda(2\delta^{ij} L_{\alpha\beta} + \epsilon_{\alpha\beta} T^{ij}), \quad \{R^i_{\dot\alpha}, R^j_{\dot\beta}\} = \lambda(2\delta^{ij} N_{\dot\alpha\dot\beta} + \epsilon_{\dot\alpha\dot\beta} T^{ij}); \tag{9}$$

$$\{Q^i_\alpha, R^j_{\dot\beta}\} = \delta^{ij}\mathscr{P}_{\alpha\dot\beta}. \tag{10}$$

Equations (7)–(10) must be supplemented with relations that express the (complex-) Lorentzian and $o(N)$ covariance of the generators, e.g.,

$$[L_{\alpha\beta}, \mathscr{P}_{\gamma\dot{\delta}}] = \frac{1}{2}(\epsilon_{\alpha\gamma}\mathscr{P}_{\beta\dot{\delta}} + \epsilon_{\beta\gamma}\mathscr{P}_{\alpha\dot{\delta}}), \quad [T^{ij}, Q^k_\alpha] = \delta^{jk}Q^i_\alpha - \delta^{ik}Q^j_\alpha . \tag{11}$$

In this algebra we can introduce the involution (Hermitian conjugation)

$$L^+_{\alpha\beta} = -N_{\dot{\alpha}\dot{\beta}}, \; N^+_{\dot{\alpha}\dot{\beta}} = -L_{\alpha\beta}, \mathscr{P}^+_{\alpha\dot{\beta}} = -\mathscr{P}_{\beta\dot{\alpha}}, (T^{ij})^+ = -T^{ij},$$

$$(Q^i_\alpha)^+ = iR^i_{\dot{\alpha}}, \; (R^i_{\dot{\beta}})^+ = iQ^i_\beta , \tag{12}$$

which corresponds precisely to the singling out of its real form osp $(N, 4; \mathbb{R})$. For osp $(N, 4, \mathbb{C})$, however, there is also an antiautomorphism*:

$$L^*_{\alpha\beta} = -N_{\dot{\alpha}\dot{\beta}}, \; N^*_{\dot{\alpha}\dot{\beta}} = -L_{\alpha\beta}, \mathscr{P}^*_{\alpha\dot{\beta}} = \mathscr{P}_{\beta\dot{\alpha}}, \; (T^{ij})^* = -T^{ij}, \; (Q^i_\alpha)^* = iR^i_{\dot{\alpha}}, \; (R^i_{\dot{\beta}})^* = -iQ^i_\beta , \tag{13}$$

which has properties (3), where $B = (L, N, \mathscr{P}, T)$, and $F = (Q, R)$. In dealing with an arbitrary representation of osp $(N, 4, \mathbb{C})$, the operation * in (13) can be associated with the Hermitian conjugation + in (5) with $K = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$. In osp $(N, 4, \mathbb{C})$, this conjugation does not correspond to an involution, since the operators KQ and KR do not belong to it.

The key point, however, is that if we switch from the operators $L, N, \mathscr{P}, T, Q, R$ to the operators $l_{\alpha\beta} = L_{\alpha\beta}$, $n_{\dot{\alpha}\dot{\beta}} = N_{\dot{\alpha}\dot{\beta}}$, $t^{ij} = T^{ij}$, $P_{\alpha\dot{\beta}} = K\mathscr{P}_{\alpha\dot{\beta}}$, $q^i_\alpha = Q^i_\alpha$, $r^i_{\dot{\beta}} = KR^i_{\dot{\beta}}$, we find that (first) these operators form the closed algebra

$$[P_{\alpha\dot{\beta}}, P_{\gamma\dot{\delta}}] = 2\lambda^2(\epsilon_{\alpha\gamma}n_{\dot{\beta}\dot{\delta}} + \epsilon_{\dot{\beta}\dot{\delta}}l_{\alpha\gamma}); \tag{14}$$

$$\{P_{\alpha\dot{\beta}}, q^i_\gamma\} = \lambda\epsilon_{\alpha\gamma}r^i_{\dot{\beta}}, \quad \{P_{\alpha\dot{\beta}}, r^i_{\dot{\delta}}\} = \lambda\epsilon_{\dot{\beta}\dot{\delta}}q^i_\alpha , \tag{15}$$

$$\{q^i_\alpha, q^j_\beta\} = \lambda(2\delta^{ij}l_{\alpha\beta} + \epsilon_{\alpha\beta}t^{ij}), \quad \{r^i_{\dot{\alpha}}, r^j_{\dot{\beta}}\} = -\lambda(2\delta^{ij}n_{\dot{\alpha}\dot{\beta}} + \epsilon_{\dot{\alpha}\dot{\beta}}t^{ij}); \tag{16}$$

$$[q^i_\alpha, r^j_{\dot{\beta}}] = -\delta^{ij}P_{\alpha\dot{\beta}}, \tag{17}$$

[the relations containing l, n, and t are analogous to relations (11)] and (second) it follows from (5) and (13) that

$$l^+_{\alpha\beta} = -n_{\dot{\alpha}\dot{\beta}}, \; n^+_{\dot{\alpha}\dot{\beta}} = -l_{\alpha\beta}, P^+_{\alpha\dot{\beta}} = P_{\beta\dot{\alpha}}, \; (t^{ij})^+ = -t^{ij}, q^{+i}_\alpha = -r^i_{\dot{\alpha}}, r^{+i}_{\dot{\alpha}} = -q^i_\alpha . \tag{18}$$

It is easy to see that the operation + in (18) specifies an involution of algebra (14)–(17), which we denote below as $\widetilde{\text{osp}}$ $(N, 4; \mathbb{C})$.

An extremely interesting point is that the algebra $\widetilde{\text{osp}}$ $(N, 4; \mathbb{C})$ constructed on the basis of the Lie superalgebra osp $(N, 4; \mathbb{C})$ is not an ordinary Lie superalgebra [cf. (15) and (17)]: $\widetilde{\text{osp}}$ $(N, 4; \mathbb{C})$ is a special case of the generalized $\underbrace{Z_2 \otimes \ldots \otimes Z_2}_{n}$-gauge algebras discussed in Ref. 5 (among other places) which are defined by

$$[g_1, g_2\} = -(-1)^{\sum_{A=1}^{n} \pi_A(g_1)\,\pi_A(g_2)} [g_2, g_1\},$$

$$(-1)^{\sum_{A=1}^{n} \pi_A(g_1)\,\pi_A(g_3)} [g_1, [g_2, g_3\}\} + (1\to 2\to 3\to 1) + (1\to 3\to 2\to 1) = 0, \tag{19}$$

where $\pi_A(g)$ $(A = 1, \ldots, n)$ are the n parities of the element g, each of which takes on the values 0, 1. It is easy to show that $\widetilde{\mathrm{osp}}\,(N, 4; \mathbb{C})$ satisfies (19) and (20) with the two parities $\pi_1(g)$ and $\pi_2(g)$, which are the same as the ordinary parities of the number of spinor indices with and without a dot on the generator g.

That the algebra $\widetilde{\mathrm{osp}}\,(N, 4; \mathbb{R})$ generated by relations (14)–(18) corresponds to a supergravity with $\Lambda > 0$ can be seen simply by noting that the transition from the Hermitian $P_{\alpha\dot\beta}$ in (18) to the anti-Hermitian $\hat{P}_{\alpha\dot\beta} = iP_{\alpha\dot\beta}$ leads to a change in the sign of λ^2 in (14) and that $\lambda^2 \sim -\Lambda$, as follows, for example, from the approach to supergravity proposed in Refs. 6–8. We wish to stress that this approach can be applied directly to $\widetilde{\mathrm{osp}}\,(N, 4; \mathbb{R})$, and the result is a supergravity with $\Lambda > 0$, as discussed above, in which the fields h, ψ, and ϕ are replaced by the fields iKh, ψ, and $K\phi$. It should be kept in mind, however, that the algebra $\widetilde{\mathrm{osp}}\,(N, 4; \mathbb{R})$ is again an expanded anti-de Sitter Lie algebra o (3, 2), and the change in the sign of Λ results from the unusual coupling of the conjugation and the involution in $\widetilde{\mathrm{osp}}\,(N, 4{:}\mathbb{C})$, which results in the Hermitian nature of $P_{\alpha\dot\beta}$.

We note in conclusion that these results seem to indicate an interesting new possibility involving the construction of supersymmetry theories based on generalized Lie superalgebras.

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Added note. After this paper had been prepared for publication, we learned that supergravity theories with $\Lambda > 0$ have also been discussed in some other recent papers[9,10] (I thank R. E. Kallosh and A. A. Tseĭtlin for bringing these papers to my attention.) In this connection, it should be emphasized that the method proposed in the present letter, which is based on the modified adjoint law * and generalized Lie superalgebras, is markedly different from the method used in Refs. 9 and 10, which is based on the ordinary adjoint and ordinary Lie superalgebras (that approach does not, however, eliminate the difficulties associated with the indefinite metric). The difference in methods can be seen most clearly in the fact that it is possible to construct supergravity theories with $\Lambda > 0$ which correspond to an N-expanded anti-de Sitter supergravity for any N in the range $1 \leqslant N \leqslant 8$, not only for even N, as in Refs. 9 and 10.

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