Random walk in an inhomogeneous medium

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It is demonstrated within the framework of perturbation theory that a two-dimensional walk in a weakly nonuniform medium reduces to pure diffusion. The nondiffusion behavior in a walk in a strongly inhomogeneous medium observed in a numerical experiment by E. Marinari *et al.* [Phys. Rev. Lett. **50**, 1223 (1983)] which is attributable to the existence of small, deep traps, is not related to the dimensionality of the space.

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We shall examine the problem of the random walk of a particle on a lattice with dimension d. At each lattice point i, the probabilities $P_{i\to j}$ of hops to the neighboring points j are random quantities $\Sigma_j P_{i\to j} = 1$ which are fixed for each point. It is known that in one dimension, such a walk turns out to be strongly non-Gaussian; the average displacement depends on the number of hops (or time) as $R(N) \propto \ln^2 N$.[1] In two dimensions, the situation is not clear. In their paper Marinari *et al.*[2] present results of a computer simulation of such a random walk and they show that depending on the degree of inhomogeneity, the random walk for large N reduces either to free (diffusion) or nondiffusion $R(N) \propto N^{\alpha}$, $\alpha < 1/2$ or $R(N) \propto \ln N$ behavior.

To describe a random walk in a weakly inhomogeneous medium, we shall use a Gaussian diagram technique, developed by Obukhov and Peliti[3] (see also Ref. 4). The vector $b_i = \Sigma_j P_{i\to j} e_{ij}$ (e_{ij} is a vector from point i to the nearest neighboring points) is introduced at each lattice point i. A random walk in a weakly inhomogeneous medium can be viewed as pure diffusion with a superposition of displacements with magnitude b_i on it at each point traversed. In the continuous limit, we obtain the following expression for the correlation function of such a walk:

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$$G(0,0;\ \mathbf{r},t) = \int D\phi D\phi^* \phi\ (0,\ 0)\ \phi(\mathbf{r},t) \exp\ \{-(H_0 + H_I)\}\ , \tag{1a}$$

$$H_0 = \int_0^\infty dt \int dr\, \phi^*(\mathbf{r},t) [- \frac{\partial}{\partial t} + D^{(0)}\ \nabla^2] \phi(\mathbf{r},t), \tag{1b}$$

$$H_I = \int_0^\infty dt \int d\mathbf{r}\, \phi(\mathbf{r},t) (1 - e^{\mathbf{b}(\mathbf{r}) \vec{\nabla}}) \phi^*(\mathbf{r},t). \tag{1c}$$

We assume that $\mathbf{b}(\mathbf{r})$ is a random function with correlation function

$$< \mathbf{b}\ (\mathbf{r}) \mathbf{b}(\mathbf{r}') > = 2d\gamma^{(0)}\ \delta\ (\mathbf{r} - \mathbf{r}'). \tag{2}$$

Expanding (1a) in powers of H_I can calculating the average using Wick's theorem, it is easy to verify that we obtain a perturbation series for the problem of a random walk with successive inclusion of one inhomogeneity, two inhomogeneties, and so on. After averaging over all possible fields $\mathbf{b}(\mathbf{r})$, we obtain, after limiting the analysis to the quadratic term in the gradients, the following expression:

$$H_I = -\gamma^{(0)} \int_0^\infty dt\, dt' \int d\mathbf{r} d\mathbf{r}'\, \phi(\mathbf{r},t) \vec{\nabla} \phi^*(\mathbf{r},t) \phi(\mathbf{r}',t') \vec{\nabla} \phi^*(\mathbf{r}',t'). \tag{1d}$$

The renormalization-group equations for a walk with an interaction of the type (1d), which contains the second power of the gradient, were obtained in general form by Obukhov and Peliti.[3] For the particular case (1d) (in the papers cited,[3] this means that $g_1^{(0)} = g_3^{(0)} = -\gamma^{(0)}$; see also Ref. 4), we obtain

$$d\gamma / d\xi = -\gamma^2 / 2, \tag{3}$$

where $\xi = \ln a'/a$ is the logarithm of the ratio of the scales. The equation has a zero-charge solution $\gamma = \gamma^{(0)}/(1 + \gamma^{(0)}\xi/2)$. We note that the first-order correction with respect to γ to the quantity $D^{(0)}$, i.e., to the diffusion coefficient, is identically equal to zero, and the second-order correction has the form $-\int_0^\infty c\gamma^2 D^{-3} d\xi = -2c\gamma^{(0)} D^{(0)-3}$, where $c \sim 1$, i.e., the interaction (1d) leads only to a renormalization of the diffusion coefficient at large scales. Higher-order terms in the gradients, which we ignored in deriving (1d), also renormalize the diffusion coefficient to some extent. This renormalization occurs at small scales and it may be assumed to be included in $D^{(0)}$. Analogously, if the distribution of the random field $b(r)$ is not purely Gaussian, then this leads to terms of the form $(\phi \vec{\nabla} \phi^*)^n$, $n > 2$ in (1d) and it is easy to see they are also unimportant at large distances.

A walk in an inhomogeneous medium also reduces to a purely random walk in a space with dimension less than two: in Eqs. (3)–(5), it is only necessary to replace $\xi = \ln a'/a$ by

$$\xi = \frac{1}{\epsilon} [(\frac{a'}{a})^{\xi} - 1],\ \ \epsilon = 2 - d. \tag{4}$$

Nevertheless, it is well known[1] that for $d = 1$, a walk in an inhomogeneous medium is localized. The case $d = 1$, is singled out, since any distribution of random forces on a straight line can be described with the help of a potential V such that $b = \nabla V$. The condition that the random force be a potential force is very strong, since for each point of the space it is necessary to introduce in this case the probability of attaining it, e^V; in this case, the starting approximation (1b) cannot be used.

We shall now show how "localization" arose in the numerical experiment in Ref. 2. In the experiment, probability distributions for the transitions were generated at each lattice point: $P_{i\to j} = a_{ij}/\Sigma_j a_{ij}^K$, where a_{ij} are random numbers from the interval (0,1). For small K, the probability distribution $P_{i\to j}$ is slightly inhomogeneous and for large K, one of the probabilities $P_{i\to j}$ is close to unity and the remaining probabilities are small. We shall now examine the simplest trap, consisting of two lattice points situated next to each other. From site 1, the particle with overwhelming probability goes over to site 2 and vice versa. This means that of the numbers a_{1j}, the number a_{12} is larger than all the others. We introduce the quantity q_1, equal to the ratio a_1^*/a_{12} where a_1^* is maximum number of all the remaining $Z-1$ numbers a_{1j} (Z is the number of nearest neighbors). The probability of leaving the trap, i.e., of going to some lattice point other than point 2, is q_1^K. The number of steps N, required to leave such a trap is larger than $N_0 \sim q_1^{-K}$. The probability that $q_1, q_2 < q$, where q is some number, is calculated in an elementary way and is equal to $Z^{-1}q^{2Z-2}$. Thus traps with N exits are distributed as $F(N) \sim N^{[(-2Z-2)/K]-1}$. This function is evidently normalizable, i.e., $\int_1^\infty F(N)dN$ exits and is determined by small N. But, the average time for leaving a trap is determined by the expression

$$\overline{N} = \int_1^\infty N F(N)\, dN \Big/ \int_1^\infty F(N)\, dN \tag{5}$$

and diverges for $K \geqslant K_c = 2Z-2$. For a two-dimensional square lattice, $K_c = 6$. The number n of traps overcome is not proportional to the total number of steps N and depends on N as $N^{6/K}$ for $K=6$, $n \propto N/\ln N$). Inclusion of more complicated traps consisting of three points, four points, and so on is important only for $K \geqslant 8$, but, apparently, does not change the asymptotic behavior indicated above. The average distance taversed over N steps is $R(N) \propto N^{3/K}$ $(K>6)$ and approaches $R(N) \propto \ln N$ in the limit of large K. The spectral density of low-frequency noise generated by such a walk $S(f) \propto f^{(-6/K)-1}$ approaches $1/f$ in the limit $K \to \infty$. In this limiting case, our model is analogous to McWhorter's[5] model, proposed as an explanation of noise in semiconductors. We note that the effect of a nondiffusion walk in a strongly inhomogeneous medium does not depend on the dimensionality of the space, but is related only to the local properties of the medium (in our case, it is determined by the coordination number of the lattice). Calculations with small K, performed in Ref. 2, confirm our conclusion that a walk in a weakly inhomogeneous medium is a purely random process.

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