Calculation of the functional determinant in the vacuum-explosion problem

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The probability for the decay of a metastable vacuum in the theory for a field of dimensionality $(1+1)$ is calculated.

In this letter we derive a complete explicit expression for the decay probability of a metastable or "false" vacuum. The problem of calculating the probability for this process arises in many physical theories: in cosmology, in the theory of phase transitions,[1] and in the theory of dislocations. The exponential factors for these decays were calculated in Refs. 1, but the matter has not been completely resolved.

Callan and Coleman[2] derived an expression for the coefficient of the exponential function in the form of a functional determinant. In this letter we wish to propose a method for calculating this determinant. We will also derive a closed expression in the approximation of a thin wall.

The probability for the decay of a metastable vacuum is determined by[2]

$$\frac{\Gamma}{V}=\frac{B}{2\pi}(D')^{-1/2}\exp(-B-\delta B);\; D'=\frac{\det'(-\partial^2+U''(\bar{\phi}))}{\det(-\partial^2+U''(\phi_+))}\quad . \tag{1}$$

Here $B=S(\bar{\phi})-S(\phi_+)$; $\delta B=S_i(\bar{\phi})-S_i(\phi_+)-B$, where S_i is a nucleating action. Here and below, the renormalized quantities are written without the corresponding index. In Eq. (1), $\bar{\phi}$ is the solution of the Euclidean equations of motion (bounce). In the limit of a small difference between the energy densities of the vacuums, this solution takes the form of a bubble of a stable vacuum which is separated from the metastable vacuum by a thin spherical wall of radius R. The prime on the determinant means that the zero eigenvalue is not included.

The simplest model with a metastable vacuum is described by the Euclidean action

$$S = \int d^2x \left[\frac{1}{2} (\partial\phi)^2 + U(\phi) \right]; \quad U(\phi) = \frac{\lambda}{8} (\phi^2 - a^2)^2 + \frac{\epsilon}{2a} \phi. \tag{2}$$

In this theory there are two vacuums, ϕ_+ and ϕ_-, with corresponding masses μ_+ and μ_-. All the specific calculations will be carried out in this model in the limit $\epsilon \to 0$. Taking into account terms of first order in ϵ, we can write the potential $U''(\bar{\phi})$ which determines the determinant in (1) as follows ($\lambda a^2 = 1$):

$$U''(\bar{\phi}) = 1 - \frac{3}{2} \operatorname{ch}^{-2} \frac{r-R}{2} - \frac{3\epsilon}{2a^2} \operatorname{th} \frac{r-R}{2}. \tag{3}$$

Everywhere below, the superior bar specifies the bounce, while the plus sign specifies the metastable vacuum.

To calculate the ratio of determinants D' in (1), we write it in the form of $D' = D_1' D_2$, where

$$D_1' = \frac{\det'(-\partial^2 + U''(\bar{\phi}))}{\det(-\partial^2 + 1 - V_{\text{th}})}; \quad D_2 = \frac{\det(-\partial^2 + 1 - V_{\text{th}})}{\det(-\partial^2 + U''(\phi_+))};$$

$$V_{\text{th}} = \frac{3\epsilon}{2a^2} \operatorname{th} \frac{r-R}{2}. \tag{4}$$

It can be shown that discarding V_{th} in D_1' does not affect the result within an accuracy to ϵ^2.

To evaluate the determinants we take the following approach. In a d-dimensional Euclidean space we write the ratio of the determinants of spherically symmetric operators as

$$X \equiv \frac{\det(-\partial^2 + \widetilde{U}(r))}{\det(-\partial^2 + U(r))} = \prod_{l, n_l} \left(\frac{\widetilde{E}_{l, n_l}}{E_{l, n_l}} \right)^{s_{d, l}}. \tag{5}$$

Here $s_{d, l}$ is the degree of degeneracy of the energy levels $\widetilde{E}_{l, n}$ and $E_{l, nl}$ with given angular momenta l and given radial quantum numbers n_l. Each product over n_l in (5) is the ratio of the determinants of one-dimensional Schrödinger equations. For the calculations we use the method described by Gel'fand and Yaglom,[3] who showed that

$$J_l \equiv \frac{\det(-d^2/dr^2 + C_l/r^2 + \widetilde{U}(r))}{\det(-d^2/dr^2 + C_l/r^2 + U(r))} = \frac{\widetilde{N}(0)\widetilde{N}(T) \int_0^T dr\, \widetilde{N}^{-2}}{N(0) N(T) \int_0^T dr\, N^{-2}}. \tag{6}$$

Here $r = 0$ and $r = T$ are the boundaries of the boundary-value problem, and C_l/r^2 is the centrifugal potential. The functions N and $\tilde{N}$ are arbitrary solutions, which do not vanish in the interval $(0, T)$, of the equation

$$\left(-\frac{d^2}{dr^2}+\frac{C_l}{r^2}+U(r)\right)N=0 \tag{7}$$

and the corresponding equation for $\tilde{N}$ with the potential $\tilde{U}$. To pursue the calculations we choose all pairs of functions N and $\tilde{N}$ such that $N(0)=\tilde{N}(0)$ and $\int drN^{-2}=\int dr\tilde{N}^{-2}$. This is obviously possible. We then have $J_l=\tilde{N}(T)/N(T)$, and the ratio of determinants becomes

$$X=\prod_l J_l^{s_{d,l}}=\prod_l (\tilde{N}_l(T)/N_l(T))^{s_{l,d}}. \tag{8}$$

Equation (6) is written for the unprimed determinant. To avoid the complication of the zero eigenvalue in D_1, we add to the potentials in the numerator and denominator of (4) a constant M, which will subsequently go to zero. Equation (8) then holds for D_1. Obviously, the entire difference between J_l and unity occurs near the well of potential (3). At $l \ll R^{3/2}$, the centrifugal potential above the well can be assumed constant (the error introduced in J_l by this assumption is of order l^2/R^4). The corresponding Schrödinger equation then becomes exactly solvable, and J_l can be found easily:

$$J_l=\frac{(k-1)(k-{}^1/_2)}{(k+1)(k+{}^1/_2)};\ k^2=1+M+\frac{4l^2-1}{4R^2}. \tag{9}$$

At finite M, J_l is found within an error of order $1/R^2$. In the limit as M goes to zero, this accuracy is not sufficient, because we find a family of $\tilde{E}_{l,0}$ levels which are themselves of order l^2/R^2. The accuracy can be improved by replacing these $\tilde{E}_{l,0}$ by their correct values $\bar{E}_{l,0}$ in product (6). As was shown in Ref. 2, the $\bar{E}_{l,0}$ can be found easily by requiring that $\bar{E}_{1,0}$, which corresponds to the zeroth mode, be equal to zero; we then find $\bar{E}_{l,0}=(l^2-1)/R^2$. The correction to J_l from (6) reduces to a multiplication by $\bar{E}_{l,0}/\tilde{E}_{l,0}$. Now setting $M=0$, we find the final result

$$J_l=\frac{(k-1)(k-1/2)}{(k+1)(k+1/2)}\,\frac{4\,(l^2-1)}{4\,l^2-1},\quad l\neq 1\ ;\ J_1'=1/12. \tag{10}$$

We might note that J_1' is exactly equal to the instanton determinant in 1D quantum mechanics.

It can be seen from (10) that at $l \gg R$ the quantity J_l has the asymptotic behavior $\ln J_l \approx (-3R)/l$. Since the WKBJ method starts to work at the same values of the angular momentum, this behavior persists at arbitrarily large l, and the product over the angular momenta diverges. This is one of the ultraviolet divergences.

We introduce a dimensional regularization. The number of dimensions affects the shape of the centrifugal potential, the degree of degeneracy $s_{d,l}$, the action in the bounce, B_d, the bubble radius R_d, and the volume V_d of the sphere of radius T. For example,[4]

$$s_{d,l}=\frac{\Gamma(l+d)}{\Gamma(d)\,\Gamma(l+1)}-\frac{\Gamma(l+d-2)}{\Gamma(d)\,\Gamma(l-1)}. \tag{11}$$

We can now use expression (8) for D_1'. We have calculated D_1' with the necessary accuracy, reducing the product to an integral at large l.

It is a simpler matter to calculate D_2. In the first place, we do not have to deal with the difficulty of a zero eigenvalue; in the second, the WKBJ method can be used to find J_l at arbitrary values of the angular momentum.

We can thus find the ratio of determinants D' in a dimensionality $2+\Delta$:

$$D' = \pi^2 R^2 \exp\left[\frac{15\epsilon R^{2+\Delta}}{2\Delta}\left(1+\Delta\left(\gamma-\frac{23}{10}-\ln 2+\frac{\pi}{5\sqrt{3}}\right)\right)\right]. \quad (12)$$

The divergences in D' disappear upon a renormalization of the parameters of the theory. In a d-dimensional space, δB from (1) is

$$\delta B = B_d\left(\frac{d}{2}\frac{\delta\lambda}{\lambda} + 3d\frac{\delta a}{a} - (d-1)\frac{\delta\epsilon}{\epsilon}\right). \quad (13)$$

We now wish to find $\delta\lambda = \lambda_i - \lambda$, δa, and $\delta\epsilon$. The simplest approach is to work in terms of an effective potential. We know that

$$U_{eff}(\phi) = U_i(\phi) + \frac{1}{2}\ln\det\left(-\partial_d^2 + U''(\phi)\right), \quad (14)$$

where ϕ is an arbitrary constant field. We specify the contraterms by the following conditions:

$$U'_{eff}(a) = \epsilon/2a; \quad U''_{eff}(a) = 1; \quad U^{IV}_{eff}(a) = 3\lambda. \quad (15)$$

The determinant in (14) can be calculated by the method described above. From (13), (14), and (15) we find

$$\delta B = \frac{15\,BR^{\Delta}}{4\pi a^2}\frac{1}{\Delta}\left[1+\Delta\left(\gamma - 7/5 - \ln 2\right)\right]. \quad (16)$$

Using (12) and (16), we find the following expression for the probability for the decay of a metastable vacuum:

$$\frac{\Gamma}{V} = \frac{\epsilon}{2\pi}\exp\left[-B + \frac{B}{\pi a^2}\left(\left(\frac{3}{2}\right)^3 - \frac{\pi\sqrt{3}}{4}\right)\right]. \quad (17)$$

A thorough analysis of the terms which we have discarded shows that the error in the argument of the exponential function is of order ϵ^{α}, where $0<\alpha<1$. Consequently, a generalization of the method to more dimensions will require more complicated calculations.

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