Relativistic equations for spin-1/2 particles

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(Submitted 10 November 1983)
Pis'ma Zh. Eksp. Teor. Fiz. **39**, No. 2, 75–78 (25 January 1984)

Two-component equations are derived for spin-1/2 particles. The Hamiltonian and momentum operators are parametrized by means of variables which arise in a determination of the basis wave functions and eigenvalues of the invariants of the fundamental series of unitary representations of the Lorentz group.

A Shapiro transformation or a Fourier transformation on the Lorentz group, which was first carried out in Ref. 2, can be generalized to a Fourier transformation on the Poincaré group if one determines not only the time (t) dependence as usual, but also the dependence on the x_k of a three-dimensional Euclidean space of frames of reference for the Heisenberg operators of the particles. Equations in terms of the variables t, x_k for the wave function were written in the Schrödinger picture in Ref. 1. The right sides of these equations have the Hamiltonian operator $H^{(0)}$ and the momentum operator $\mathbf{P}^{(0)}$ of a particle with spin 0 (Ref. 3):

$$H^{(0)} = \operatorname{ch} i\frac{\partial}{\partial\alpha} + \frac{i}{\alpha}\operatorname{sh} i\frac{\partial}{\partial\alpha} - \frac{\Delta_{\theta,\varphi}}{2\alpha^2}\exp i\frac{\partial}{\partial\alpha} \quad ; \tag{1}$$

$$\mathbf{P}^{(0)} = -\mathbf{n}\left(e^{i\frac{\partial}{\partial\alpha}} - H^{(0)}\right) - [\mathbf{n}\cdot\mathbf{L}(\theta,\varphi)]\frac{1}{\alpha}\exp i\frac{\partial}{\partial\alpha} \; ; \tag{2}$$

$$0 \leqslant \alpha < \infty; \quad |\mathbf{n}| = 1; \quad \mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta).$$

The representation variables α, $\mathbf{n}$, which can be used to parametrize the states of the relativistic particles in accordance with Refs. 2–5, arise in a determination of the basis functions and eigenvalues of the invariants of the Lorentz group, $F = \mathbf{N}^2 - \mathbf{J}^2$ and $G = \mathbf{NJ}$. The operators $H^{(0)}$ and $\mathbf{P}^{(0)}$, the angular-momentum operators

$\mathbf{J}^{(0)} = \mathbf{L}(\theta,\varphi) \equiv \mathbf{L}$, and the operators

$$\mathbf{N}^{(0)}(\alpha,\mathbf{n}) = \mathbf{n}(\alpha - i) + [\mathbf{n}\cdot\mathbf{L}] \tag{3}$$

satisfy the commutation relations of the Lie algebra of the Poincaré group.[1)]

Since the differentiation operators $-i(\partial/\partial x_k)$ are not momentum operators according to Ref. 1, we want to find the Hamiltonian operator H, the momentum operator $\mathbf{P}$, the angular-momentum operator $\mathbf{J}$, and the operators $\mathbf{N}$ for spin-1/2 particles in the "α", $\mathbf{n}$ representation." In this letter we derive these operators, and we write equations for the wave functions of these particles. Previous attempts have been made (see Ref. 6, for example) to find Hamiltonian and momentum operators for particles with a spin in the α, $\mathbf{n}$ representation. The formal expressions written in Ref. 6 for these operators, however, are defined not only in terms of α and $\mathbf{n}$ but also in terms of the variables of the momentum representation—a meaningless procedure.

Working from the familiar expression $\mathbf{J} = \mathbf{L} + \vec{\sigma}/2$, where the $\vec{\sigma}$ are the Pauli matrices, we can write

$$\mathbf{N}(\alpha,\mathbf{n}) = \mathbf{n}(\alpha - i) + [\mathbf{n}\cdot\mathbf{J}]. \tag{4}$$

The commutation relations of the Lie algebra of the Lorentz group are satisfied. Furthermore, we have $F = 1 + \alpha^2 - \frac{1}{4}$. In determining H and $\mathbf{P}$ we work from the basis functions of the invariant F in the momentum representation. In this case we have[7]

$$N_k(p) = ip_0\frac{\partial}{\partial p_k} - \frac{1}{2}\frac{[\vec{\sigma}\cdot\mathbf{p}]_k}{p_0+1}\;;\; J_k(\mathbf{p}) = i\epsilon_{kij}p_j\frac{\partial}{\partial p_0} + \frac{\sigma_k}{2}, \tag{5}$$

$$p_0^2 - \mathbf{p}^2 = 1;\; (i,j,k = 1,2,3).$$

In the equation

$$F(\mathbf{p})\Psi(\mathbf{p},\alpha,\mathbf{n}) = (1 + \alpha^2 - \frac{1}{4})\Psi(\mathbf{p},\alpha,\mathbf{n}) \tag{6}$$

we single out the function $f^* = (p_0 - \mathbf{pn})^{-1+i\alpha}$. We then find the two two-component solutions

$$\Psi_{1/2}(\mathbf{p},\alpha,\mathbf{n}) = D^*\cdot f^*\cdot\chi_{1/2}\;;\quad \Psi_{-1/2}(\mathbf{p},\alpha,\mathbf{n}) = D^*\cdot f^*\cdot\chi_{-1/2}, \tag{7}$$

where

$$D^* = \frac{p_0 - \mathbf{pn} + 1 - i\vec{\sigma}\cdot[\mathbf{p}\cdot\mathbf{n}]}{\sqrt{2(p_0+1)(p_0 - p\cdot n)}}\;;\quad D^*\cdot D = 1;$$

$$\chi_{1/2} = \begin{pmatrix}1\\0\end{pmatrix};\quad \chi_{-1/2} = \begin{pmatrix}0\\1\end{pmatrix}. \tag{8}$$

We now introduce functions of the variables α, $\mathbf{n}$:

$$\Psi_{1/2}(\alpha,\mathbf{n};\mathbf{p}) = Df\chi_{1/2}\;;\quad \Psi_{-1/2}(\alpha,\mathbf{n},\mathbf{p}) = Df\chi_{-1/2}, \tag{9}$$

where $f = (p_0 - \mathbf{pn})^{-1-i\alpha}$ is an eigenfunction of the operators $H^{(0)}$ and $\mathbf{P}^{(0)}$. Using the operator

$$B = 2\,\mathrm{ch}\,\frac{i}{2}\,\frac{\partial}{\partial\alpha} - i\,\frac{\mathbf{L}\cdot\vec{\sigma}}{\alpha - \frac{i}{2}}\,\exp\frac{i}{2}\,\frac{\partial}{\partial\alpha} \tag{10}$$

we can rewrite (9) as

$$\Psi_{1/2}(\alpha, \mathbf{n}, \mathbf{p}) = B\,\frac{f}{\sqrt{2(p_0 + 1)}}\cdot \chi_{1/2}\,;$$

$$\Psi_{-1/2}(\alpha, \mathbf{n}, \mathbf{p}) = B\,\frac{f}{\sqrt{2(p_0 + 1)}}\,\chi_{-1/2}\,. \tag{11}$$

We now seek an operator K such that the following equality holds:

$$KB = 2H^{(0)} + 2\,. \tag{12}$$

Substituting $H^{(0)}$ and B into (12), we find

$$K = 2\,\mathrm{ch}\,\frac{i}{2}\,\frac{\partial}{\partial\alpha} + \frac{2i}{\alpha}\,\mathrm{sh}\,\frac{i}{2}\,\frac{\partial}{\partial\alpha} + i\,\frac{\mathbf{L}\vec{\sigma}}{\alpha - \frac{i}{2}}\,\exp\frac{i}{2}\,\frac{\partial}{\partial\alpha}\,. \tag{13}$$

From the equation

$$B\left(\frac{KB}{2} - 1\right)\frac{f}{\sqrt{2(p_0 + 1)}} = p_0 D f \tag{14}$$

we then find that the operator $(BK)/2 - 1$ is diagonal in the states (9) with eigenvalues which are equal to the energy of the particle, p_0. This is the operator H which we have been seeking. Its explicit expression is

$$H = \mathrm{ch}\,i\frac{\partial}{\partial\alpha} + \frac{i}{2\left(\alpha + \frac{i}{2}\right)}\,e^{i\frac{\partial}{\partial\alpha}} - \frac{i}{2\left(\alpha - \frac{i}{2}\right)}\,e^{-i\frac{\partial}{\partial\alpha}} + \frac{-\Delta_{\theta,\varphi}\,e^{i\frac{\partial}{\partial\alpha}} + \mathbf{L}\vec{\sigma}\left(e^{i\frac{\partial}{\partial\alpha}} - 1\right) - 1}{2\left(\alpha^2 + \frac{1}{4}\right)}. \tag{15}$$

The momentum operators $\mathbf{P}$ can be found in two ways: either by working from the commutation relations

$$[N_k(\alpha, \mathbf{n})H]_- = iP_k \tag{16}$$

or by using the operator expression

$$BP_k^{(0)} K = 2HP_k + 2P_k , \tag{17}$$

which follows from the eigenvalue problem

$$BP_k^{(0)} KB \frac{f}{\sqrt{2(p_0 + 1)}} = (2p_0 p_k + 2p_k)Df . \tag{18}$$

In either case we have

$$\mathbf{P} = - \mathbf{n}\left(e^{i\frac{\partial}{\partial\alpha}} - H\right) + \frac{-2\alpha[\mathbf{n}\cdot\mathbf{L}] - \left(\alpha - \frac{i}{2}\right)[\mathbf{n}\cdot\vec{\sigma}] - (\mathbf{n}\sigma)\mathbf{L}}{2\,\alpha^2 + \frac{1}{4}} e^{i\frac{\partial}{\partial\alpha}}$$

$$+ \frac{[\mathbf{n}\cdot\sigma]}{2\left(\alpha + \frac{i}{2}\right)} . \tag{19}$$

In addition to (16), all the other commutation relations of the Lie algebra of the Poincaré group hold for these operators.

In the Schrödinger picture the eigenfunctions (9) of the operators H and $\mathbf{P}$ would be multiplied by $\exp[-i(p_0t - \mathbf{px})]$, according to Ref. 1. As a result, we have the following generalization for the superposition of these states, in place of the Fourier transformation on the Lorentz group[4,5]:

$$\Psi(\alpha, \mathbf{n}, \mathbf{x}t) = \frac{1}{(\partial\pi)^3} \int \frac{d^3p}{p_0} Df \cdot e^{-i(p_0t - \mathbf{px})} \cdot \Psi(p), \tag{20}$$

where

$$\Psi(\mathbf{p}) = \begin{pmatrix} \Psi_1(\mathbf{p}) \\ \Psi_2(\mathbf{p}) \end{pmatrix} .$$

The following equations also hold:

$$i\frac{\partial}{\partial t} \Psi(\alpha, \mathbf{n}, \mathbf{x}, t) = H\Psi(\alpha, \mathbf{n}, \mathbf{x}t);$$

$$-i\frac{\partial}{\partial x_k} \Psi(\alpha, \mathbf{n}, \mathbf{x}, t) = P_k\Psi(\alpha, \mathbf{n}, \mathbf{x}, t). \tag{21}$$

[1)] We are using a system of units with $m = 1$, $\hbar = 1$, $c = 1$.

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Translated by Dave Parsons
Edited by S. J. Amoretty